The algebra of alternation and synchronisation of finite dynamical systems

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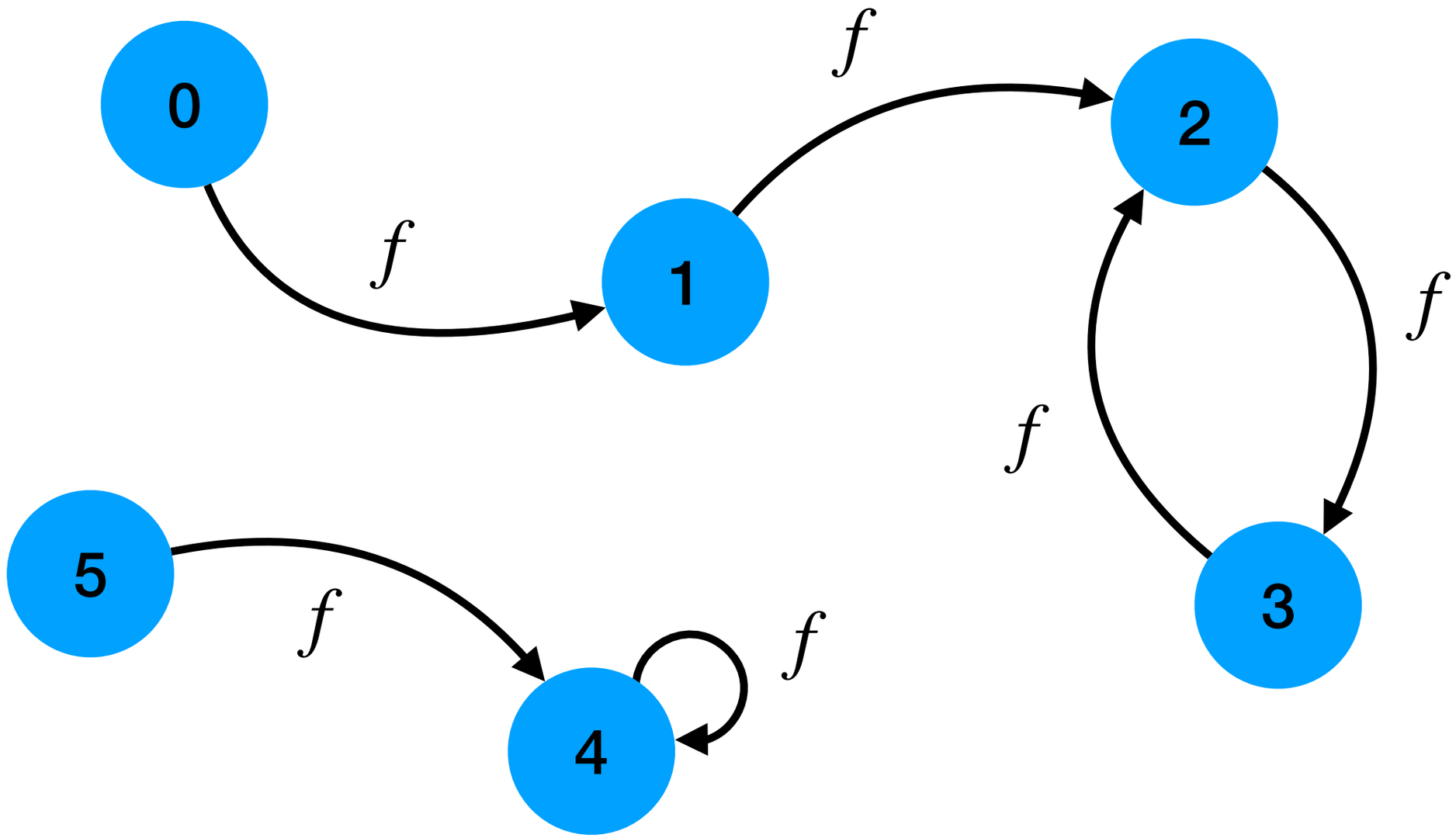
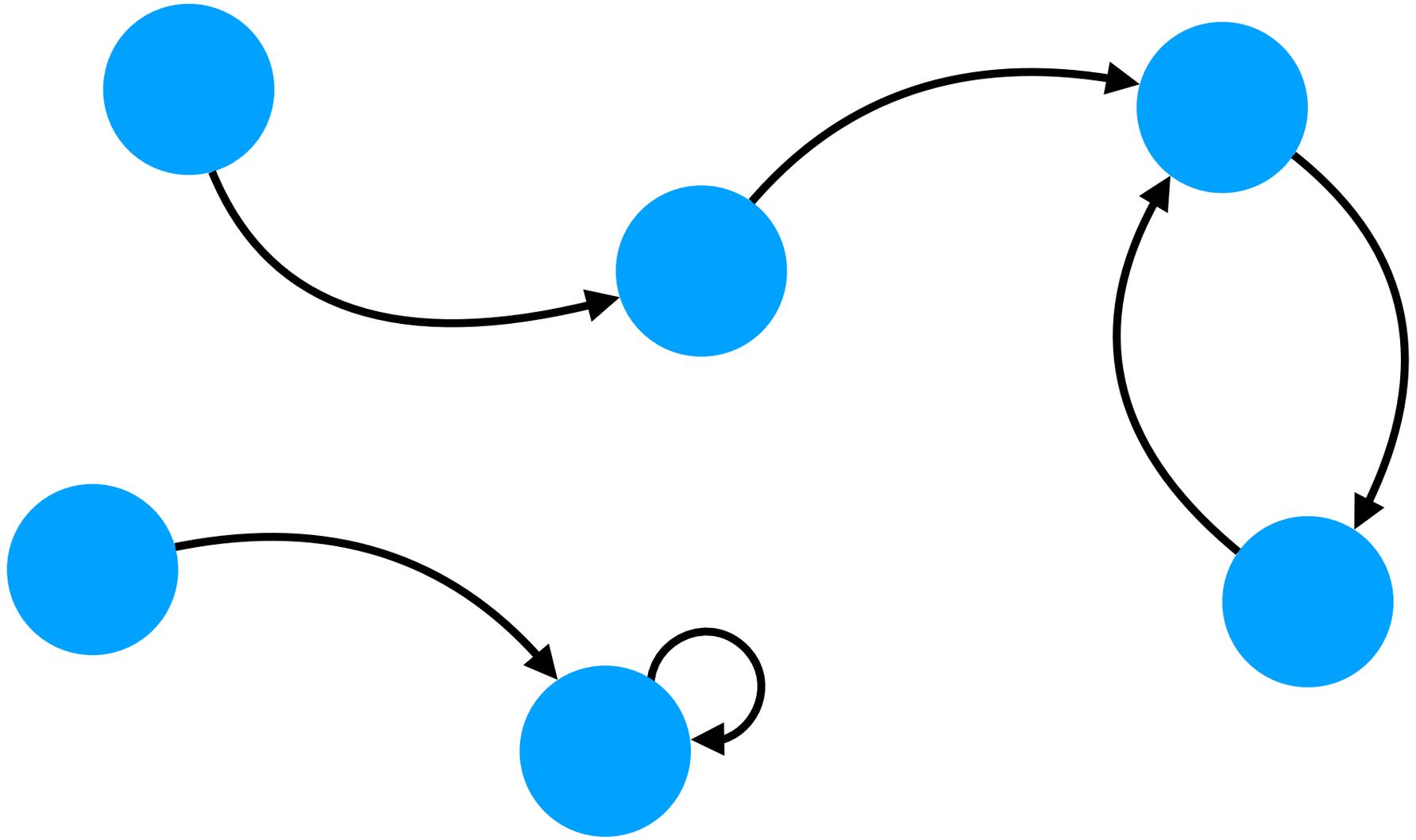
Finite, discrete-time dynamical systems

Just a finite set with a transition function (A, f)



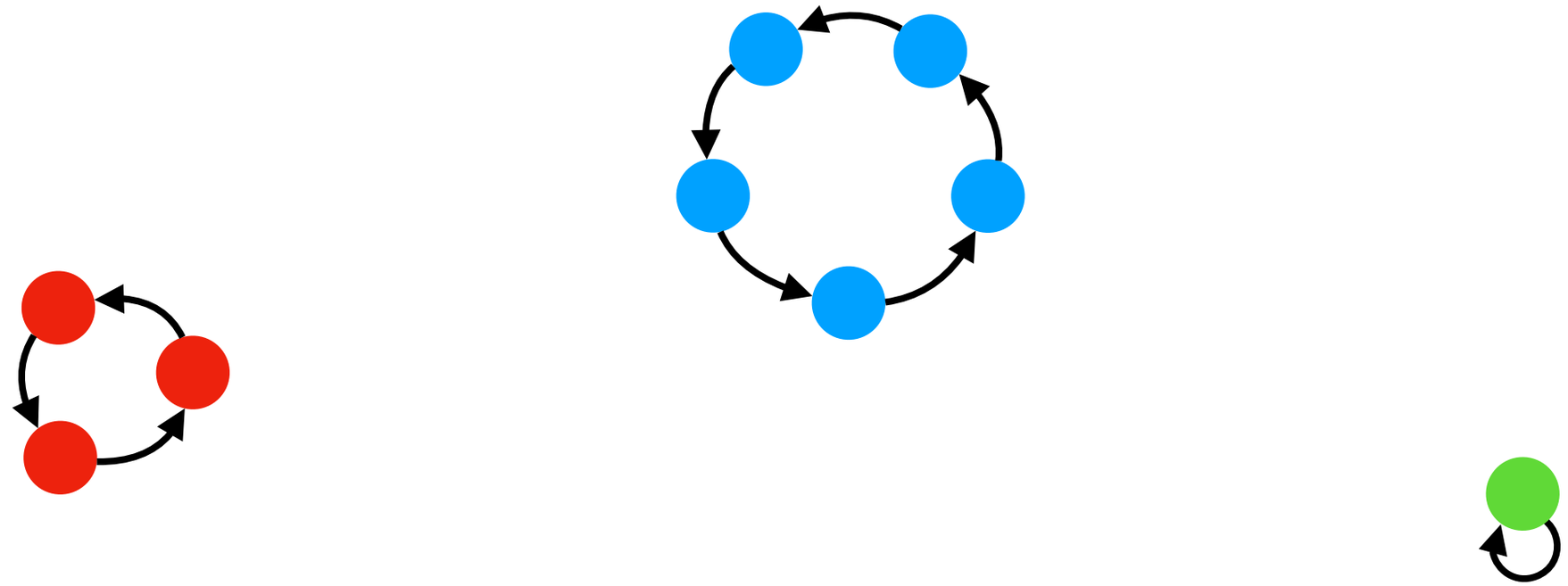
Finite, discrete-time dynamical systems

Just a finite set with a transition function (A, f) modulo isomorphism



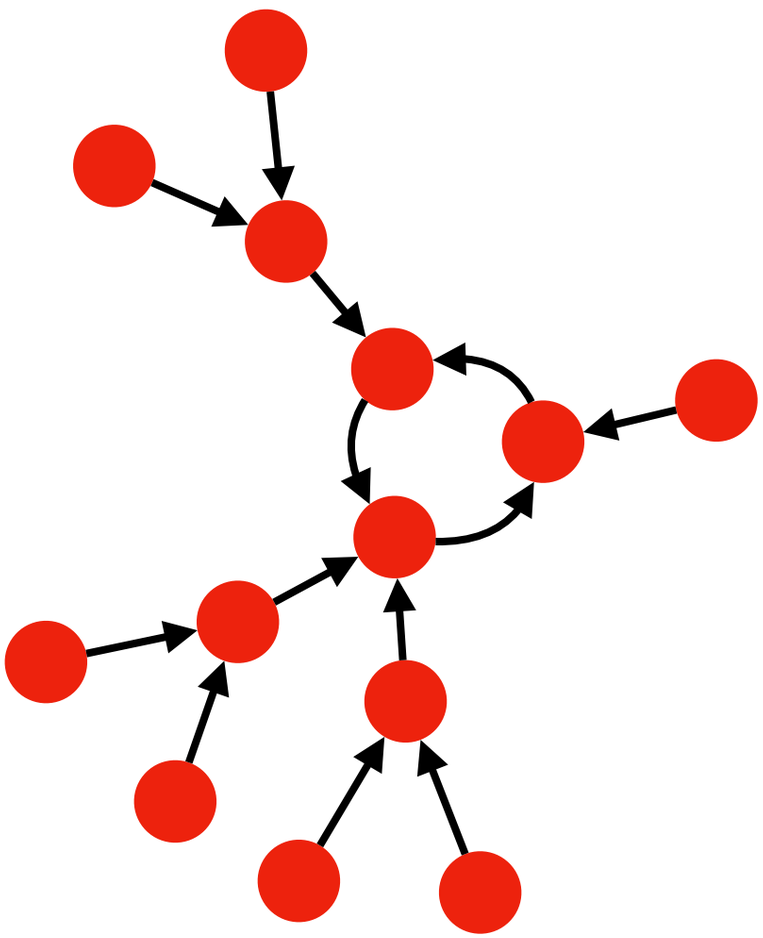
General shape of a dynamical system

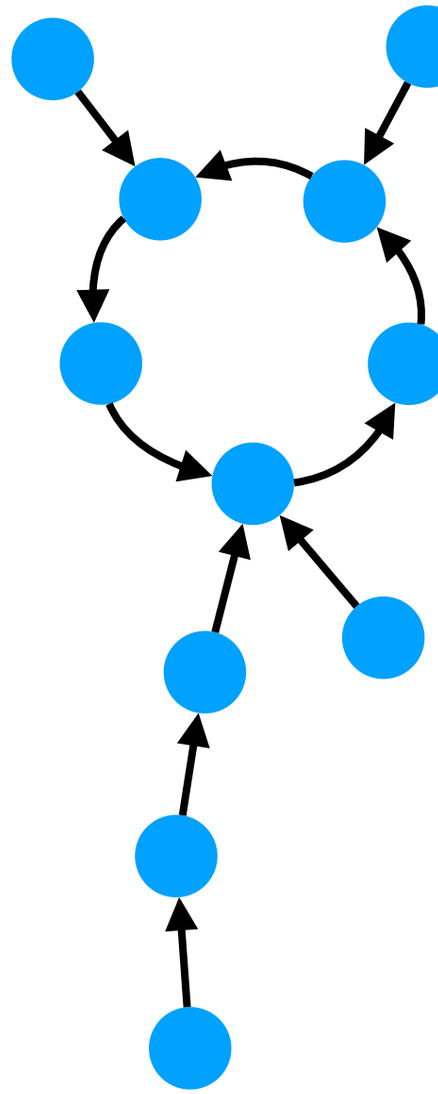
A few limit cycles

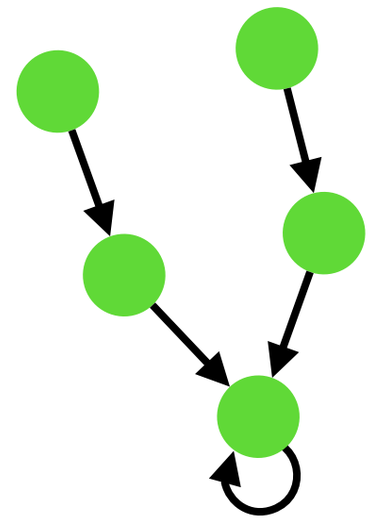


General shape of a dynamical system

A few limit cycles with trees going in







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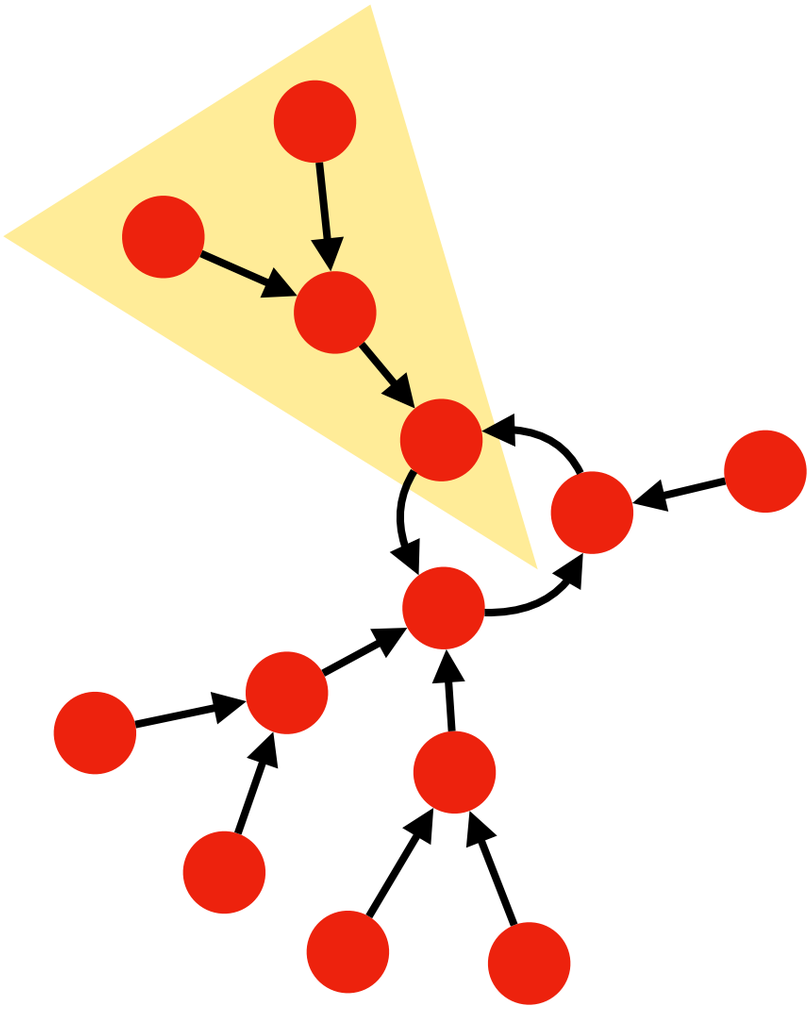
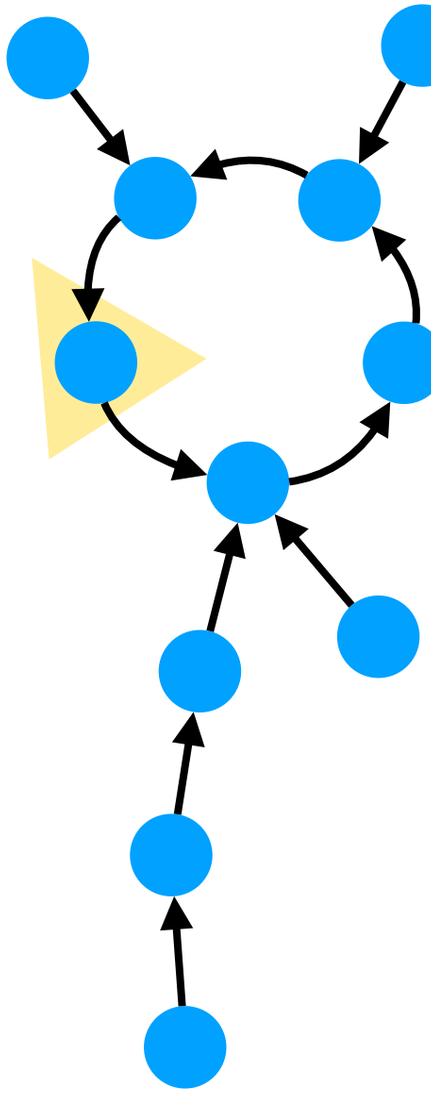
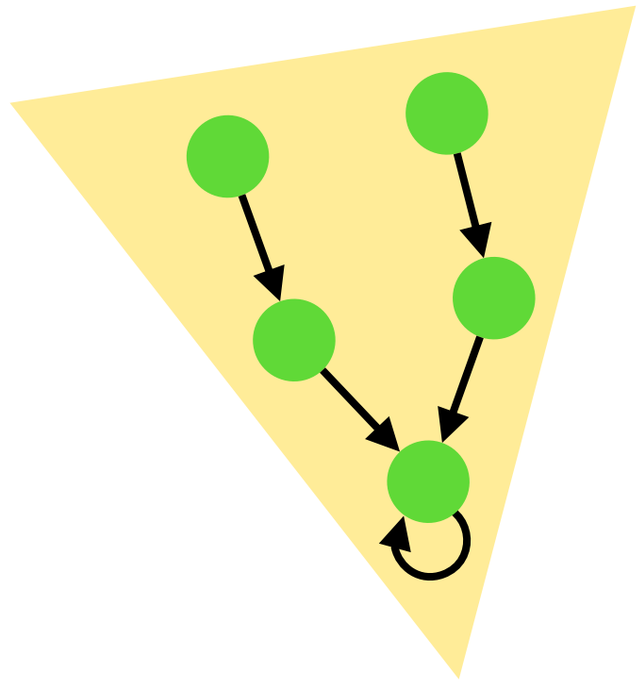
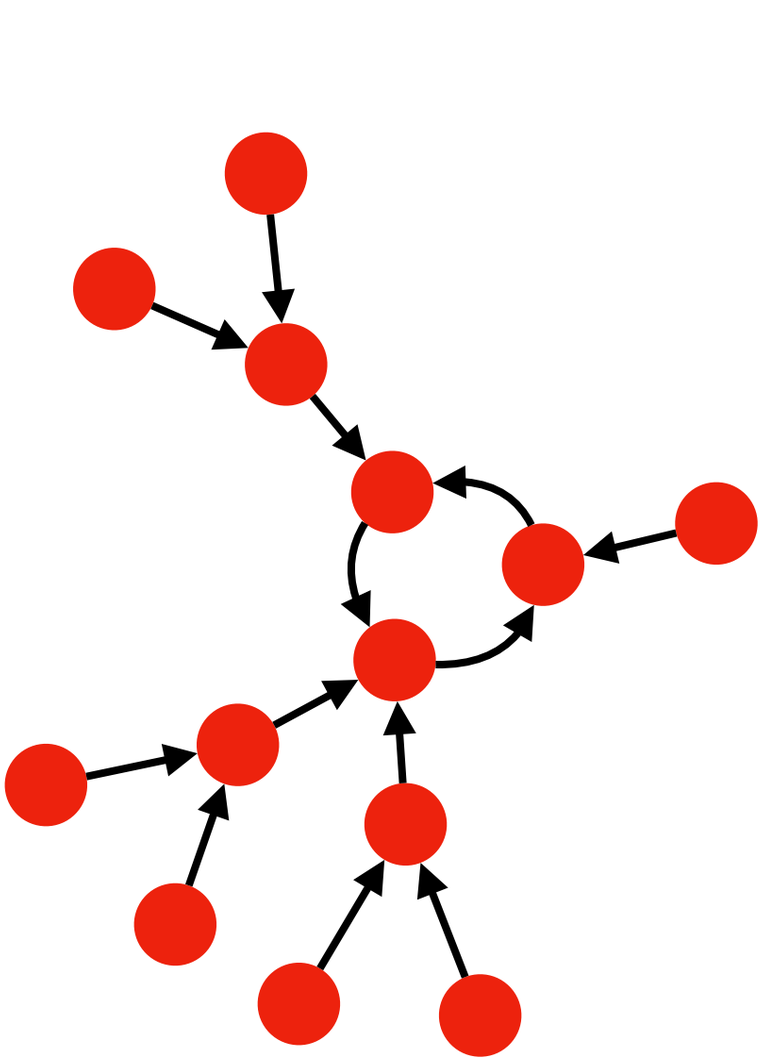


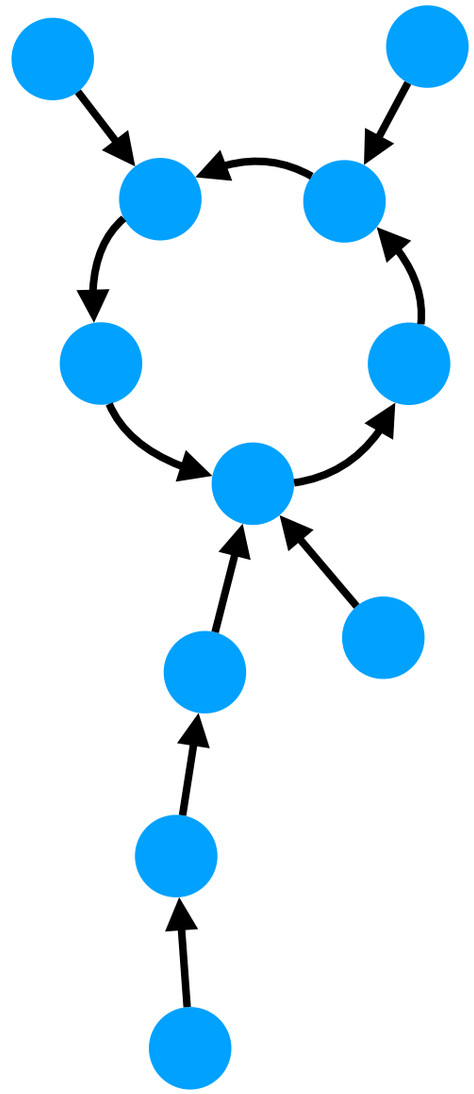


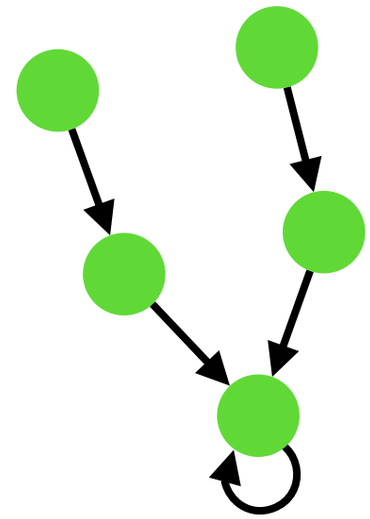


General shape of a dynamical system

A few limit cycles with trees going in







$$C_3\left(\ ,\ ,\ ,\right) \quad + \quad C_5\left(\ ,\ ,\ ,\ ,\ \right) \quad + \quad C_1\left(\ \right)$$

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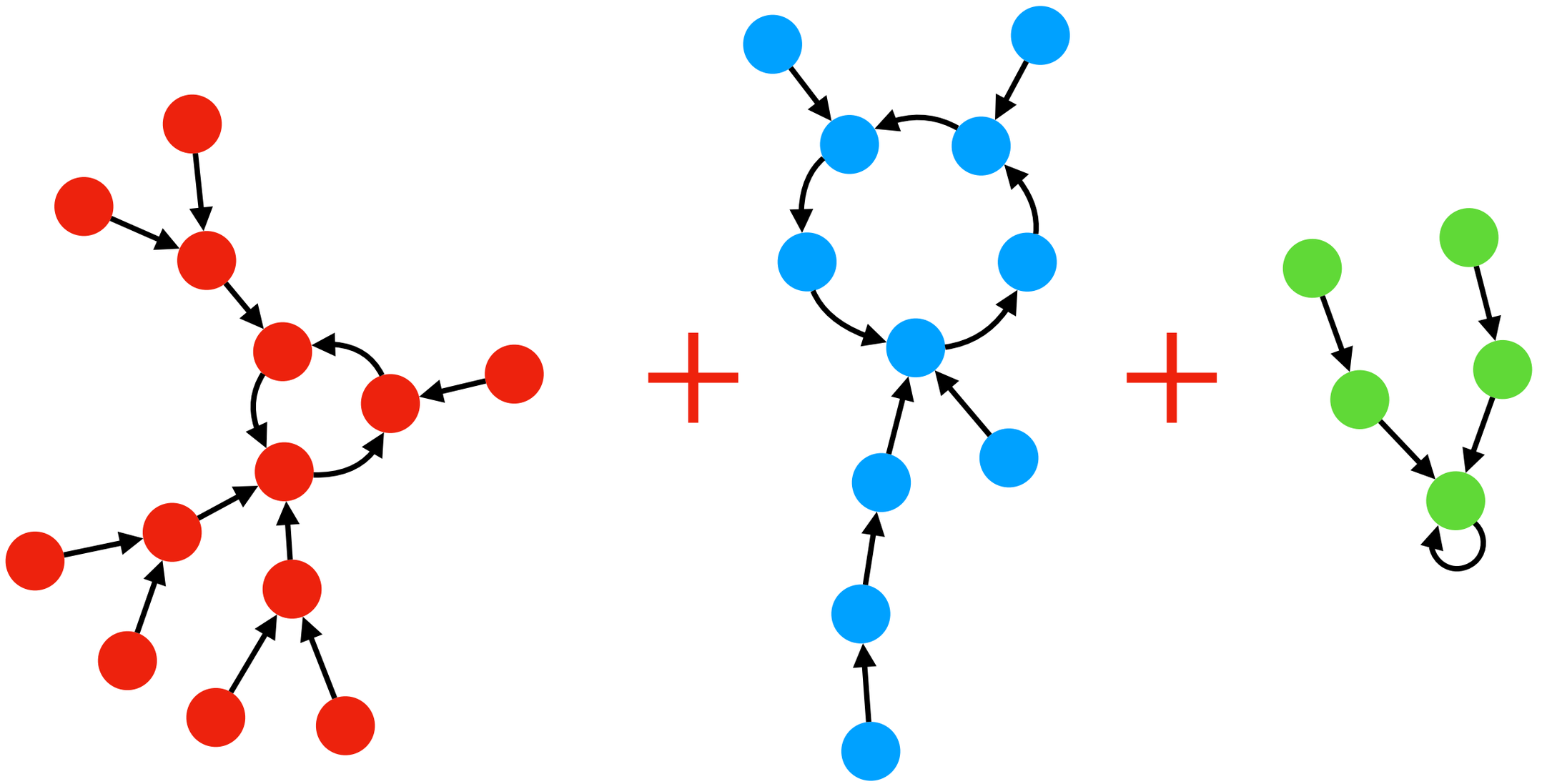

$$C_3\left(\ ,\ ,\ ,\ \right) + C_5\left(\ ,\ ,\ ,\ ,\ ,\ \right) + C_1\left(\ \right)$$

Isomorphism of dynamical systems is easy

2009 24th Annual IEEE Conference on Computational Complexity

Planar Graph Isomorphism is in Log-Space

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Email: fabian.wagner@uni-ulm.de

Abstract

Graph Isomorphism is the prime example of a computational problem with a wide difference between the best known lower and upper bounds on its complexity. There is a significant gap between extant lower and upper bounds for planar graphs as well. We bridge the gap for this natural and important special case by presenting an upper bound

The problem is clearly in NP and by a group theoretic proof also in SPP [AK06]. This is the current frontier of our knowledge as far as upper bounds go. The inability to give efficient algorithms for the problem would lead one to believe that the problem is provably hard. NP-hardness is precluded by a result that states if GI is NP-hard then the polynomial time hierarchy collapses to the second level [BHZ87], [Sch88]. What is more surprising is that not even P-hardness is known for the problem. The best we know is that GI is hard for DET [Tor04], the class of problems

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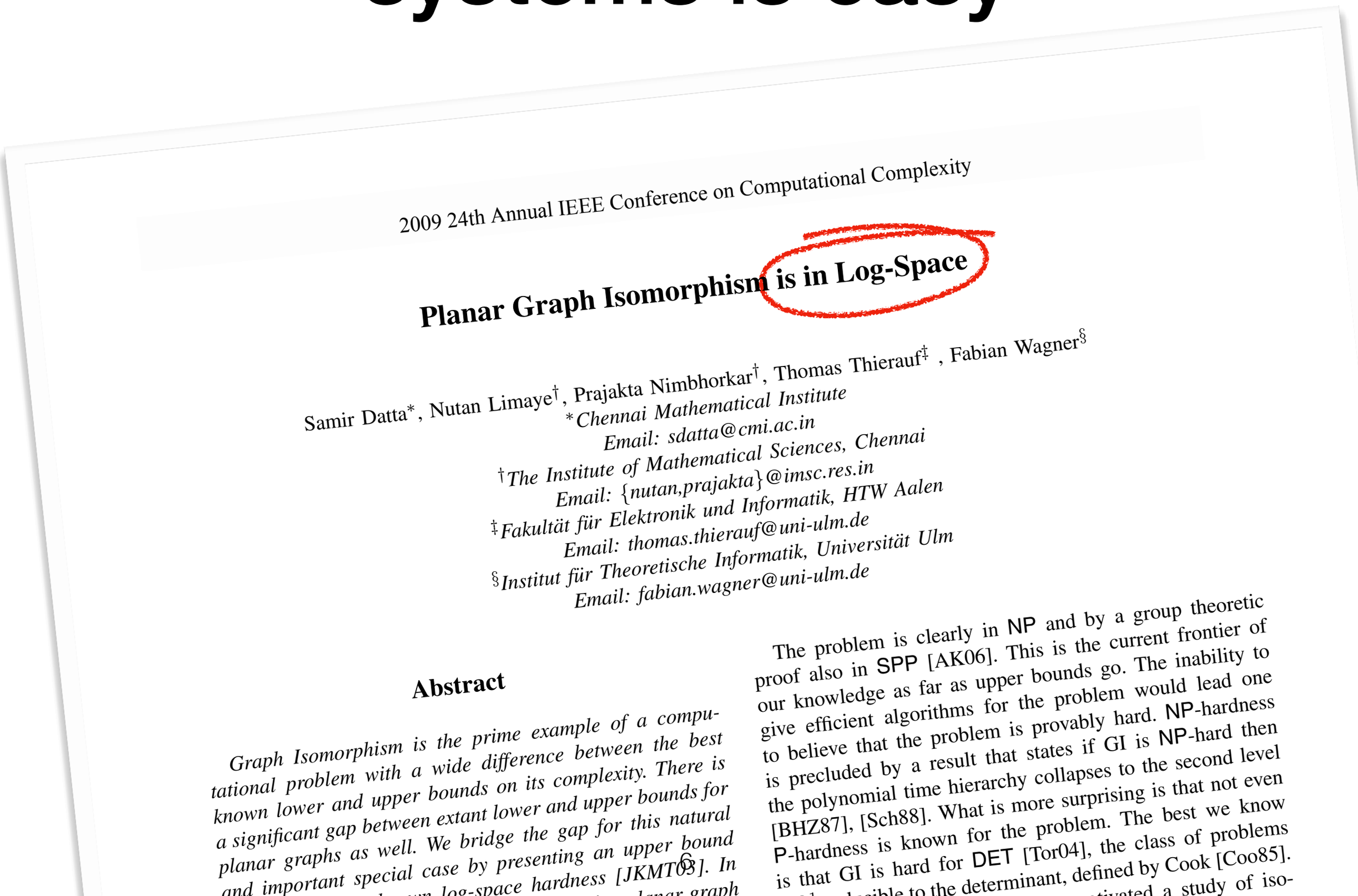

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log-space hardness [JKMT03]. In

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INTERMISSION

How to efficiently generate dynamical systems, aka functional digraphs

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- In theory: Antonio E. Porreca, Ekaterina Timofeeva, Polynomial-delay generation of functional digraphs up to isomorphism, arXiv:2302.13832

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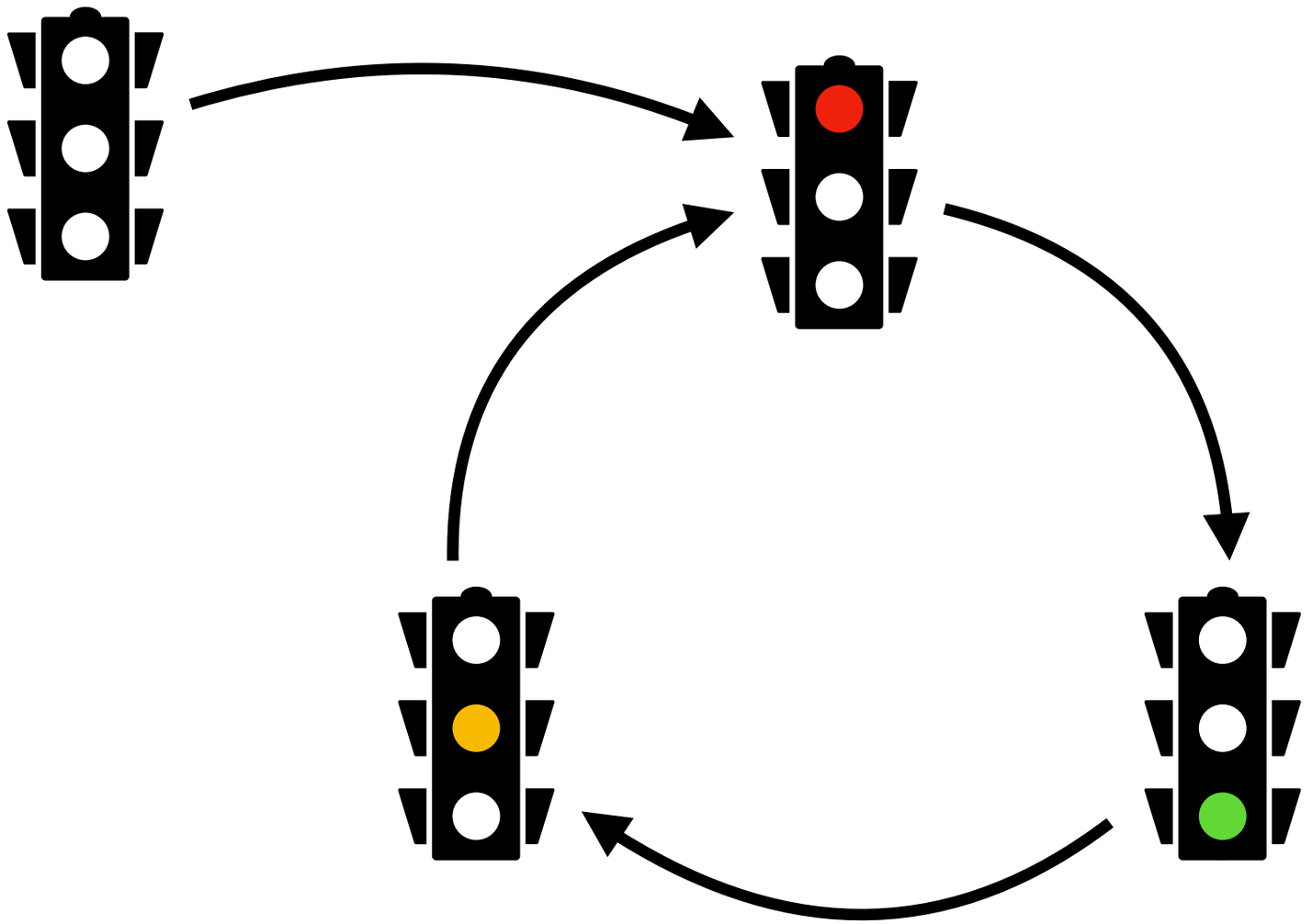
- In practice: `funkdigen2`, a fast implementation of the above, github.com/aeporreca/funkdigen2

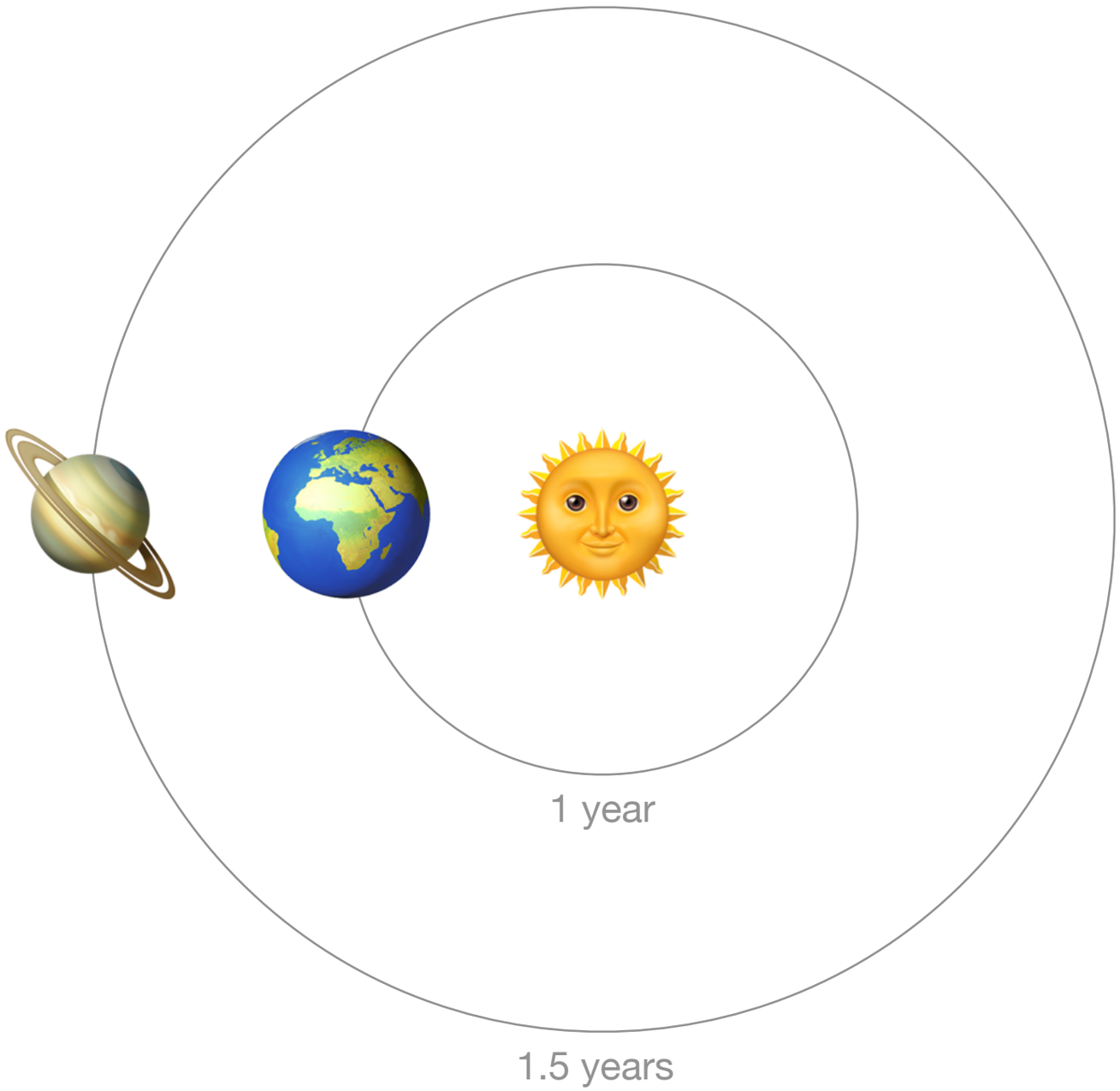
A toy example from engineering

Traffic lights

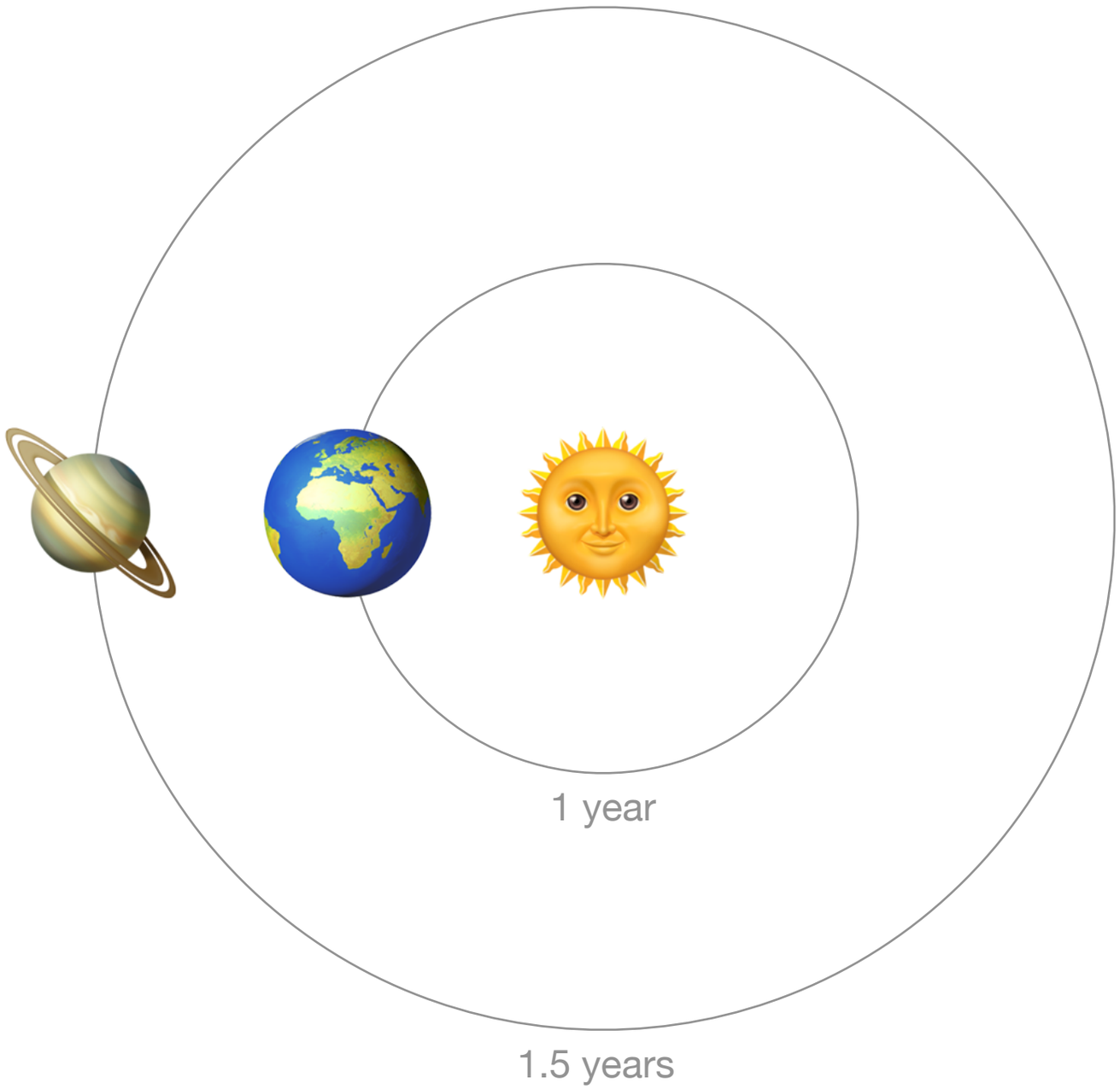


A toy example from science

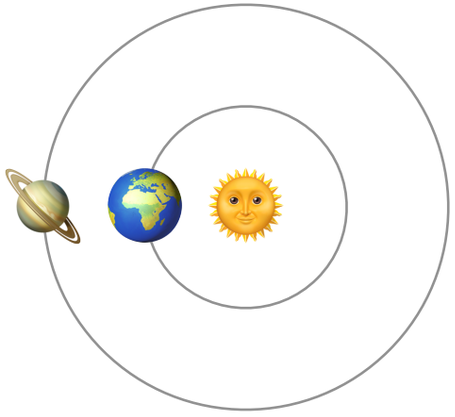
A planetary system



A planetary system



Evolution in time



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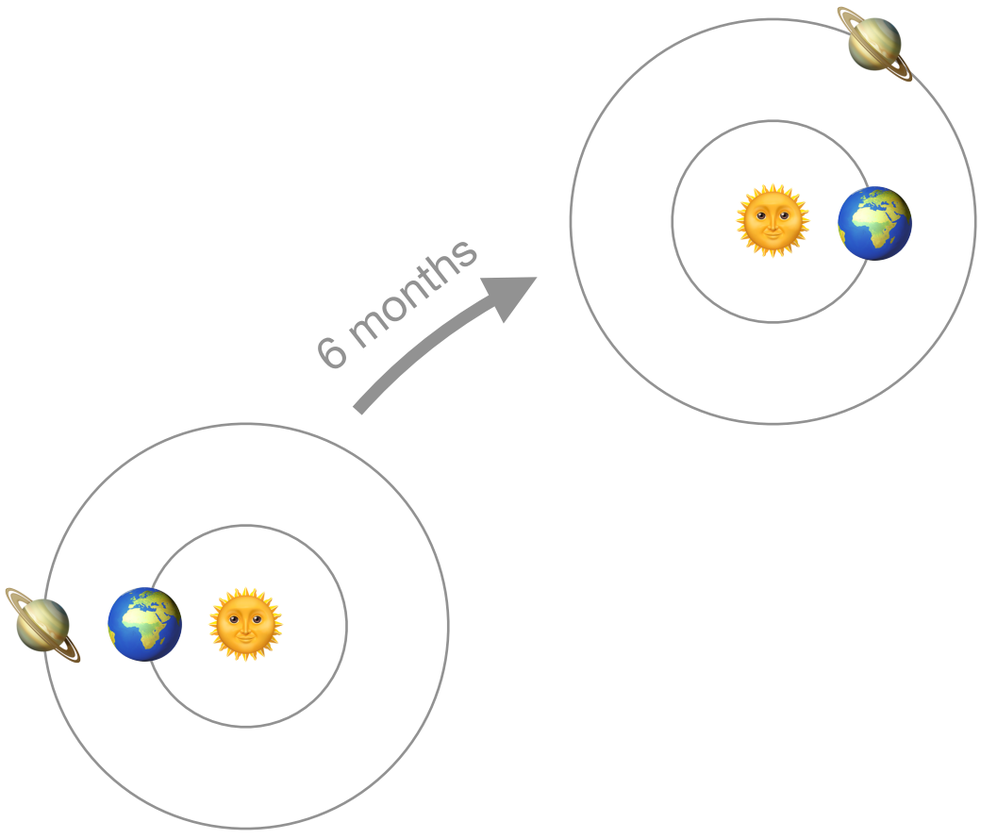

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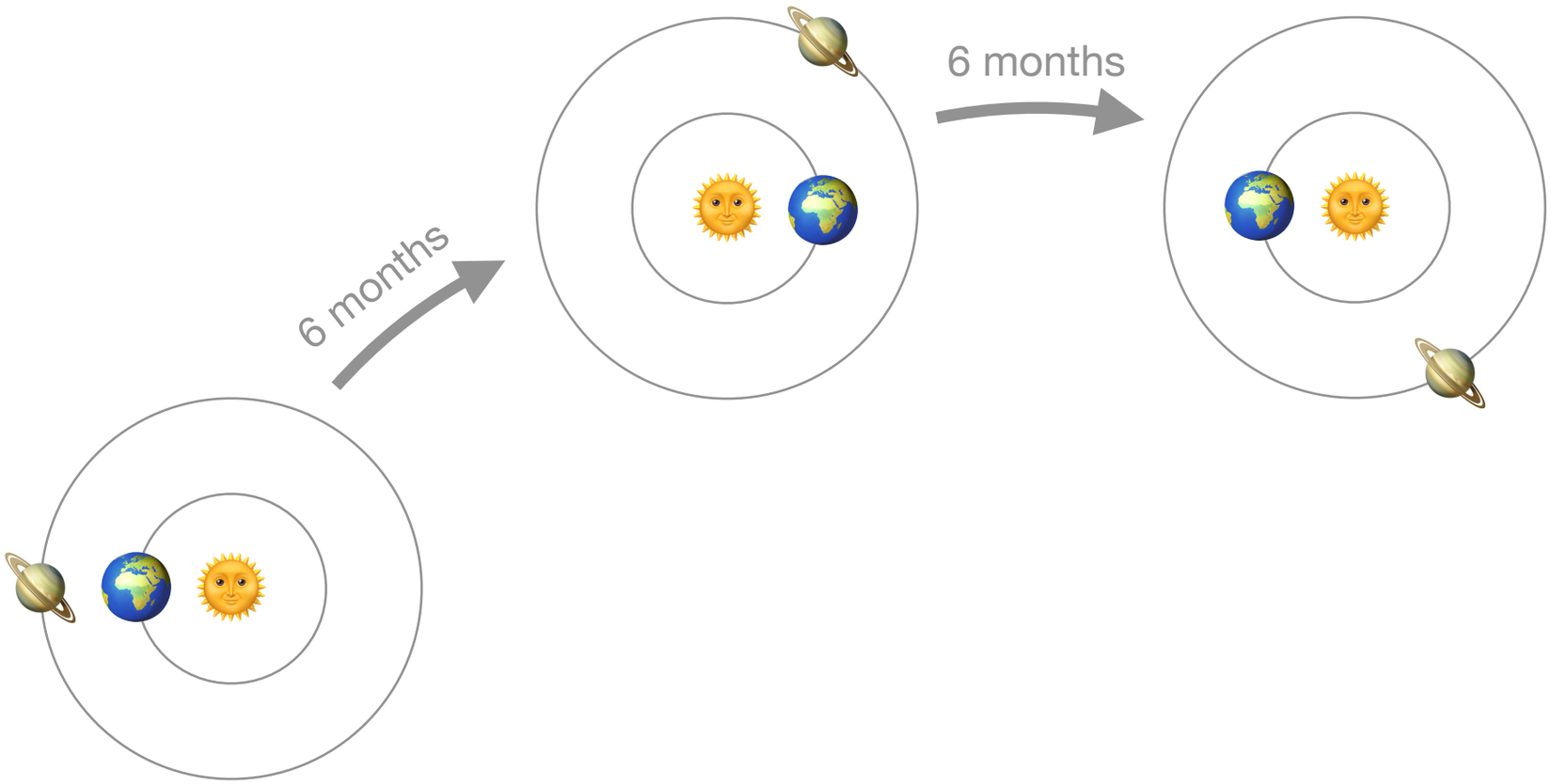

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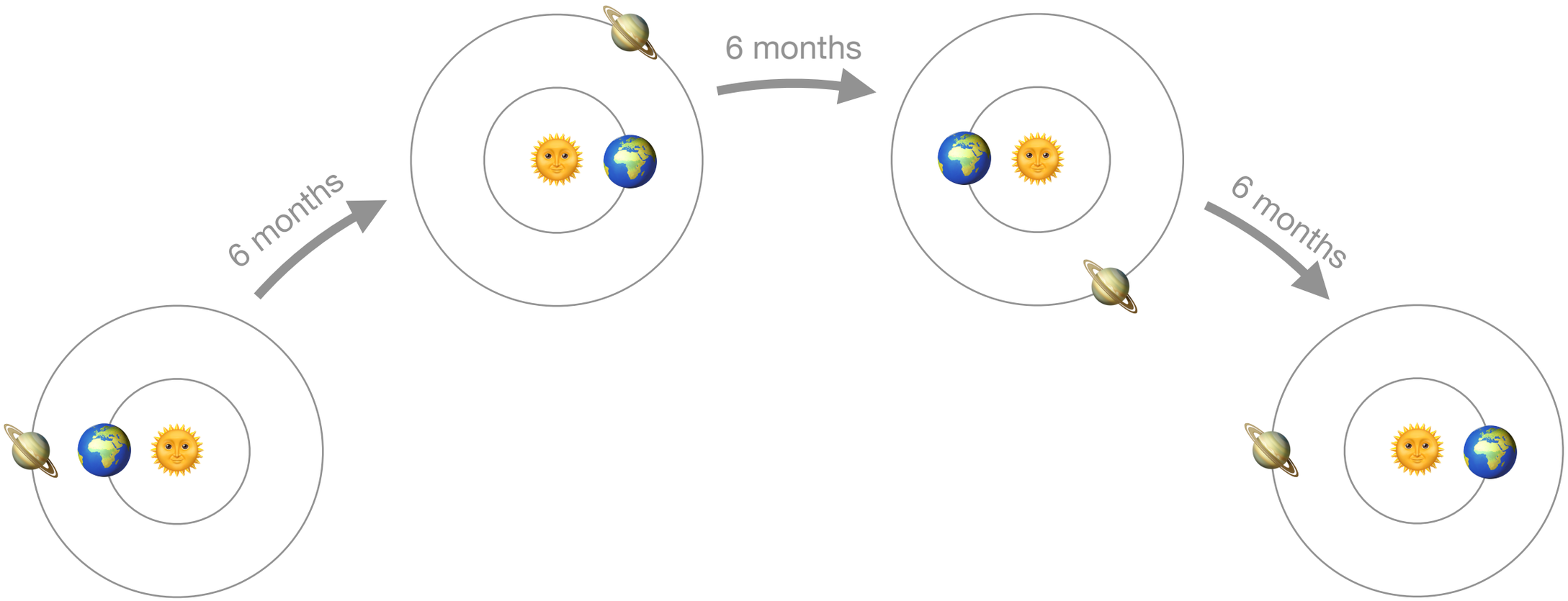

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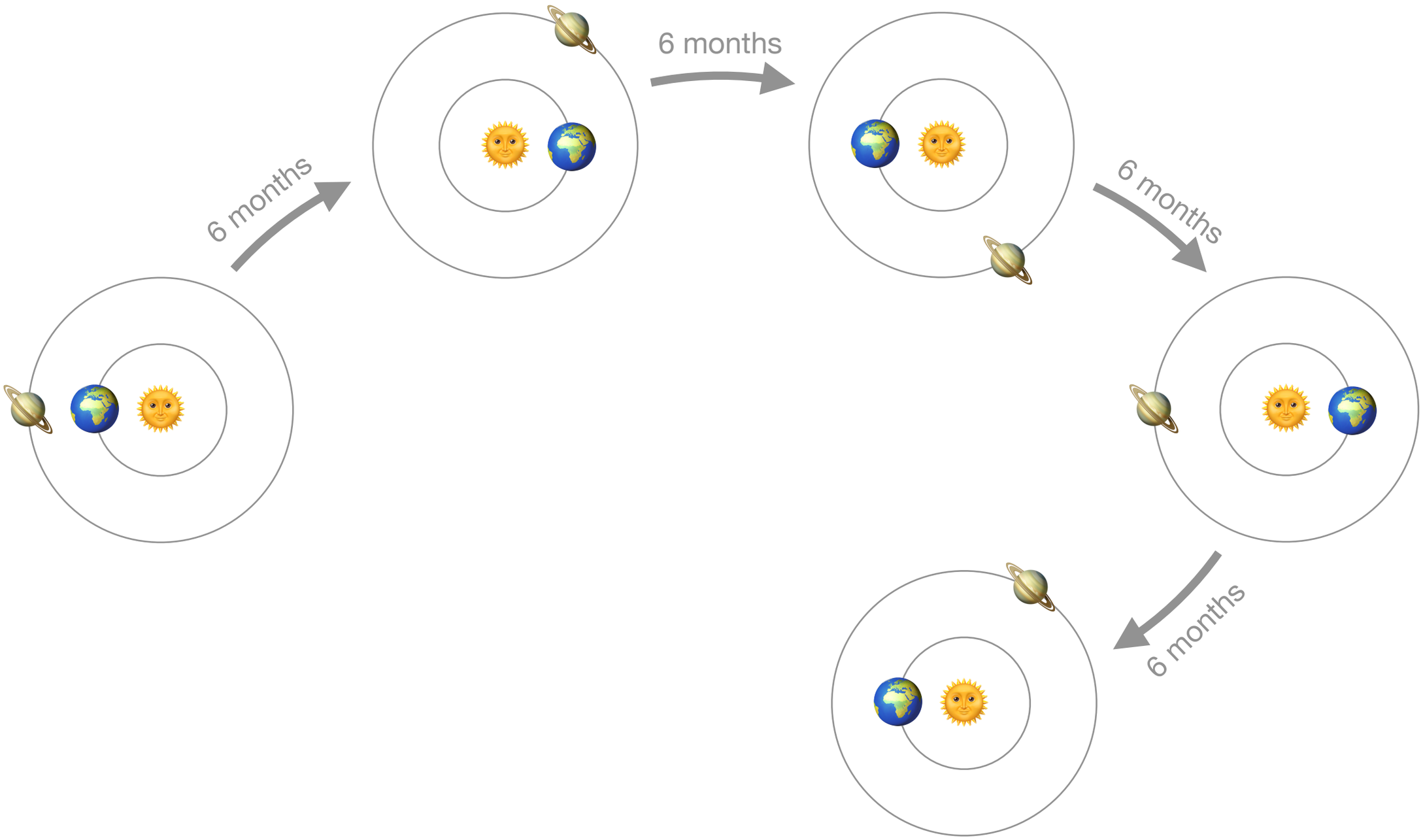

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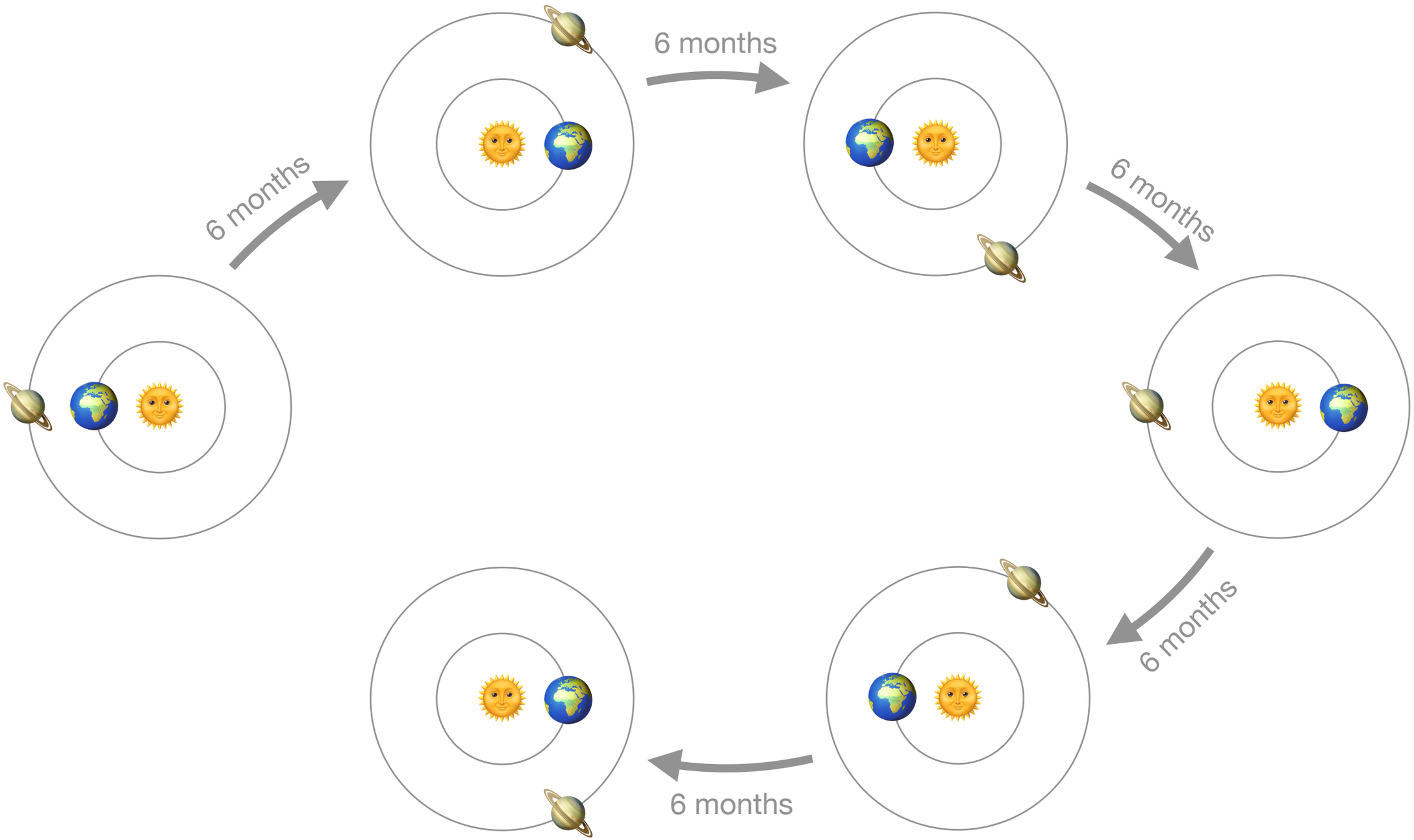

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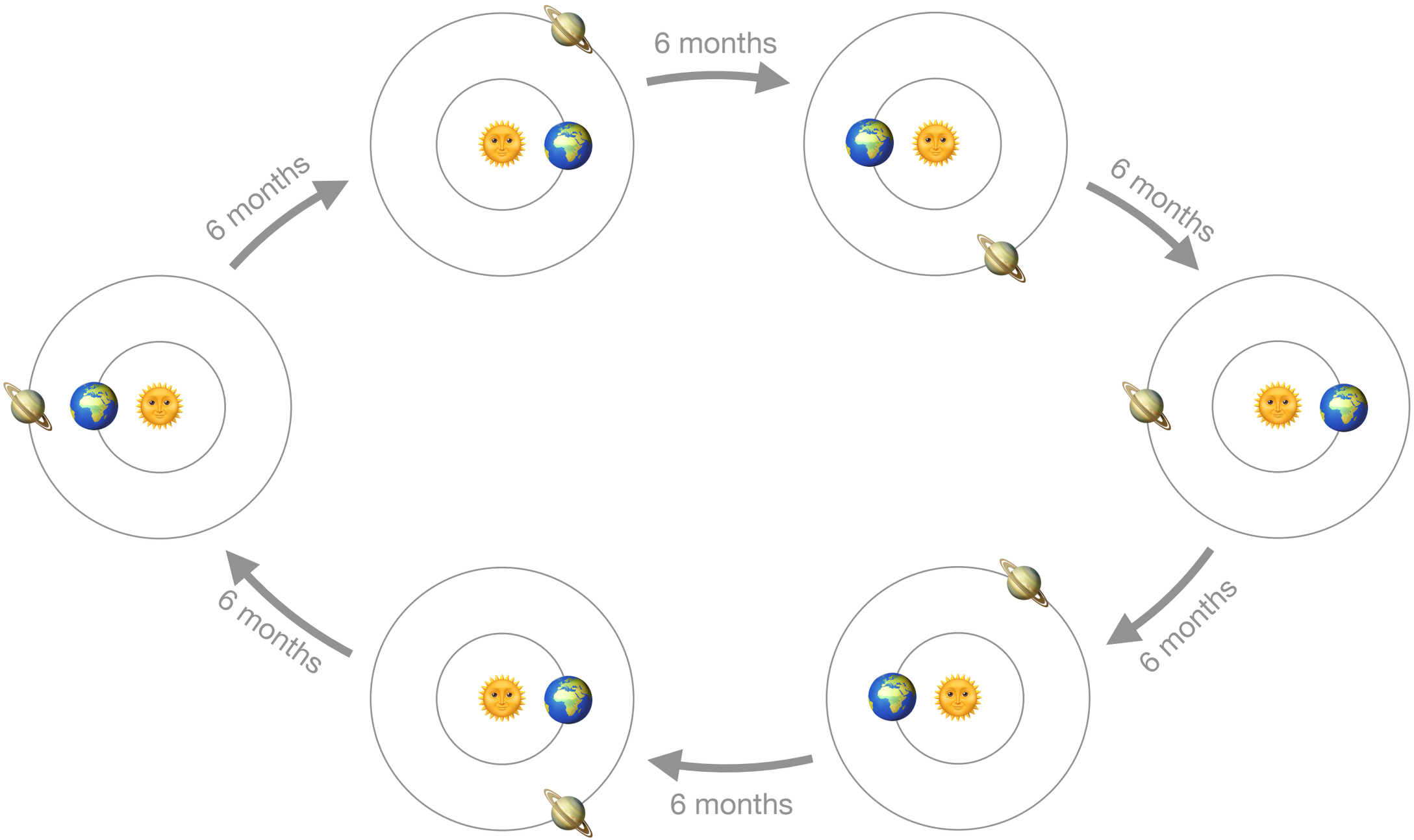

Decomposing the system

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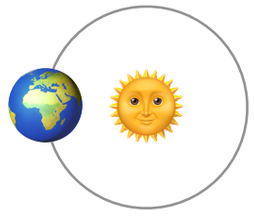

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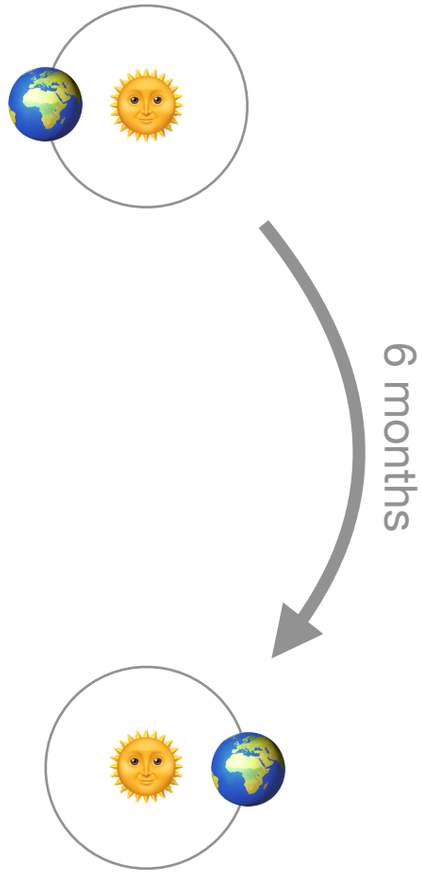

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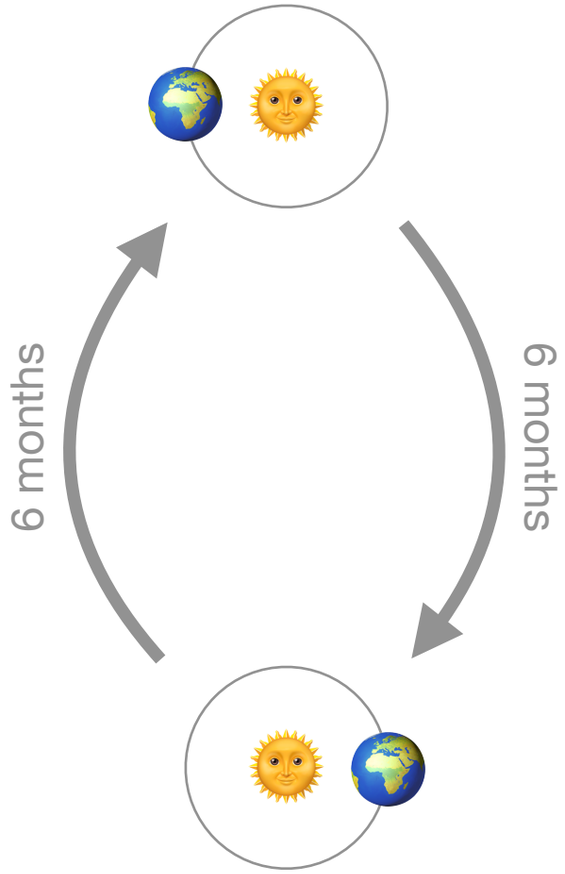

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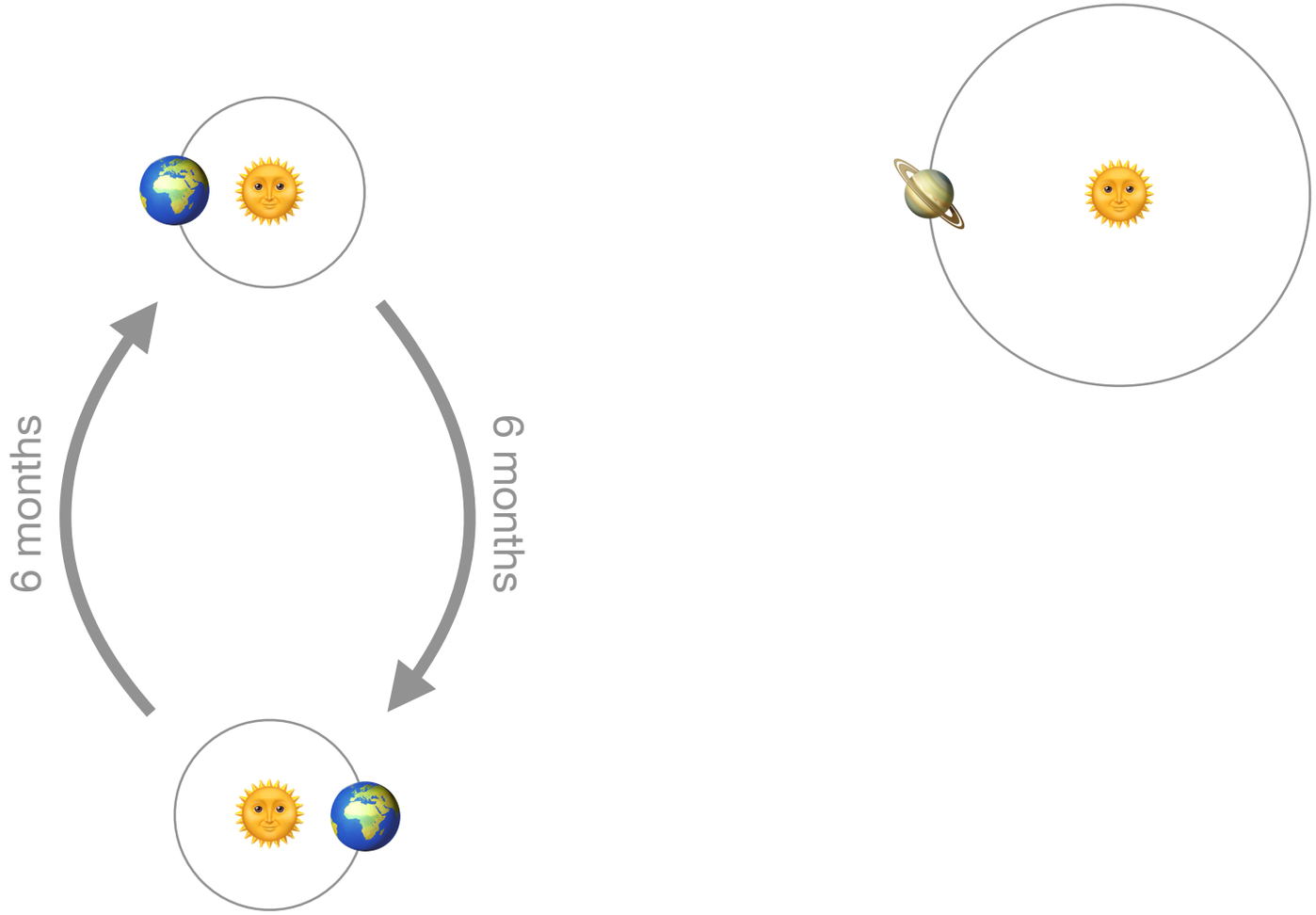

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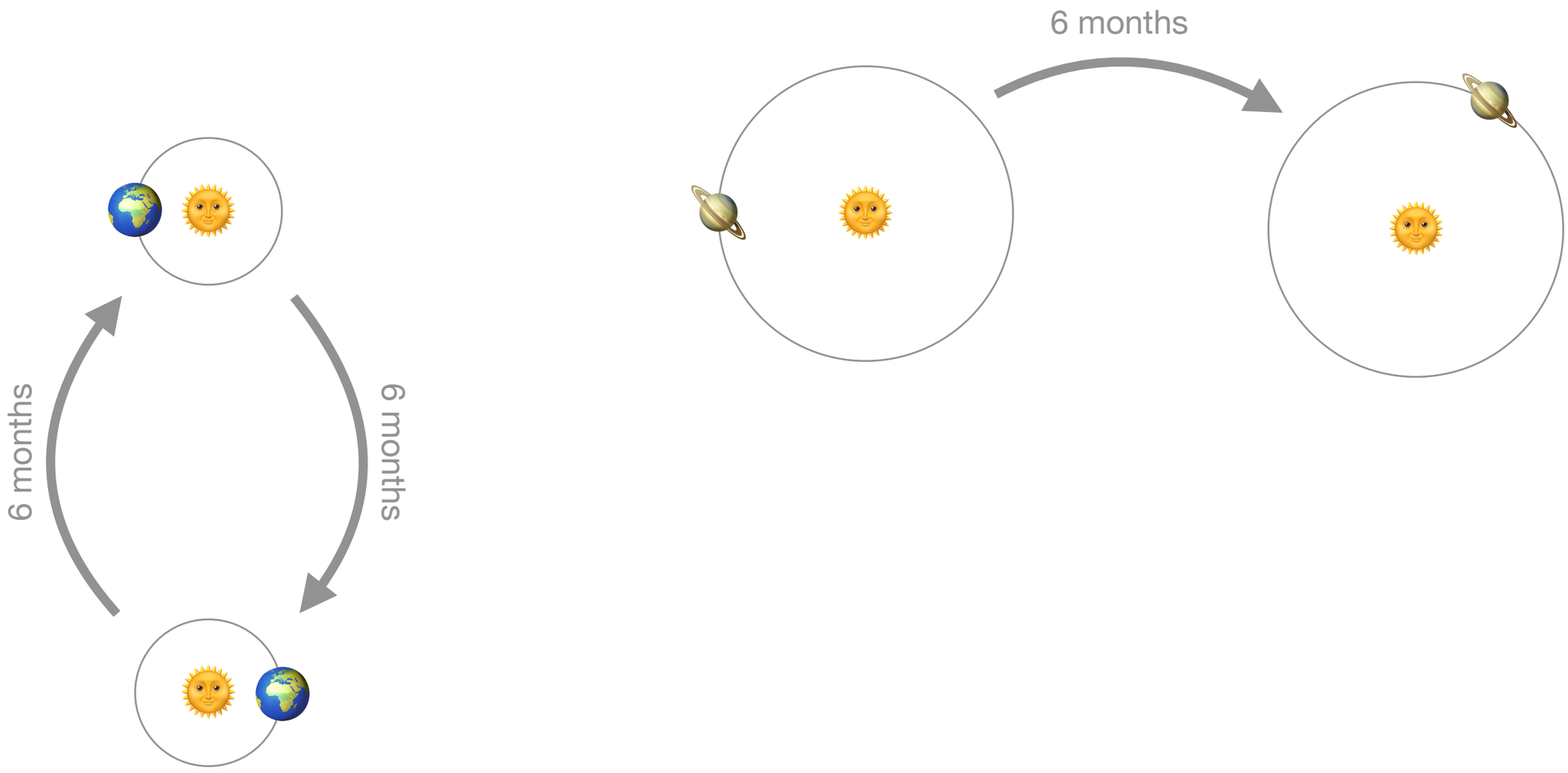

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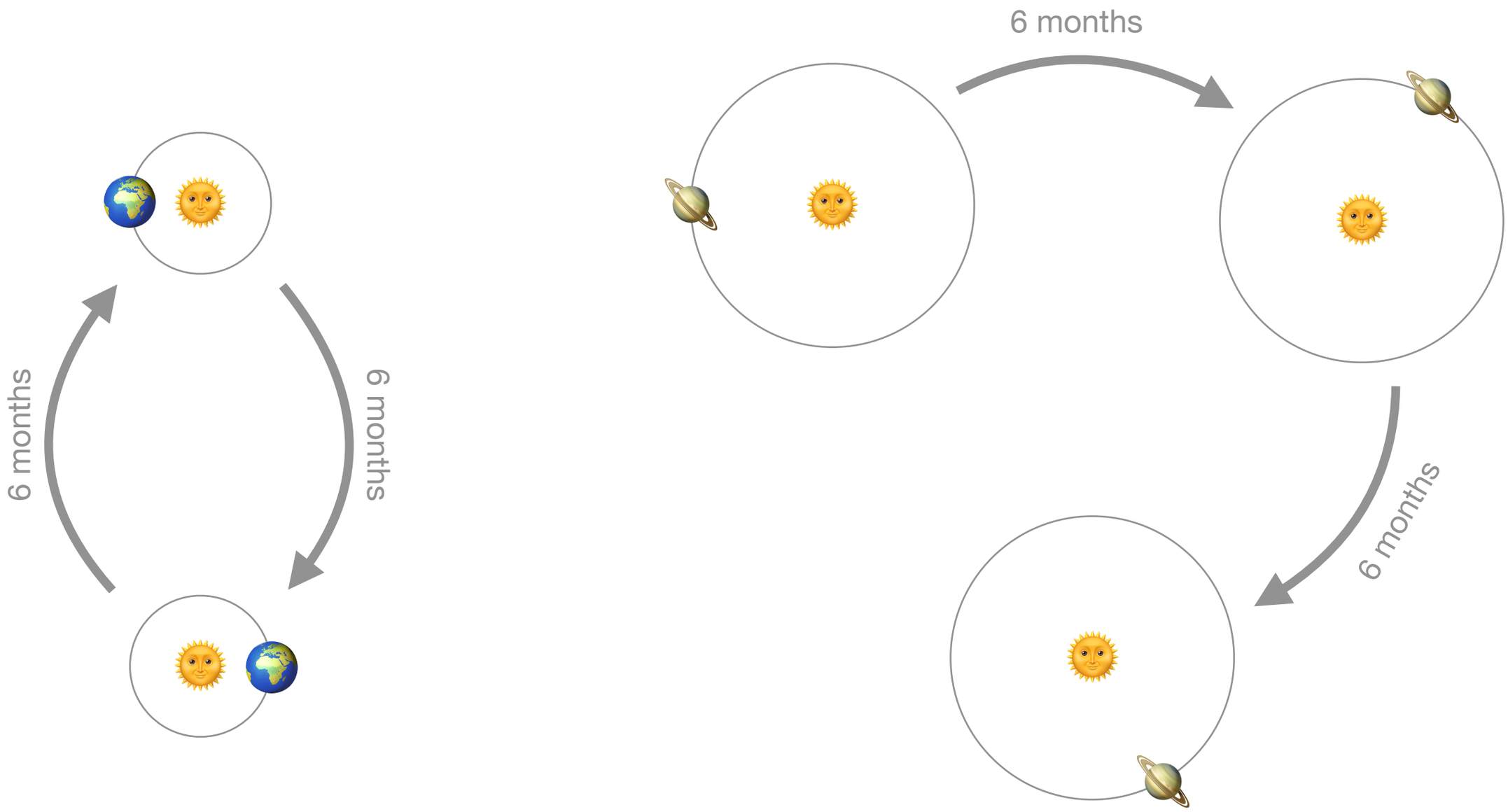

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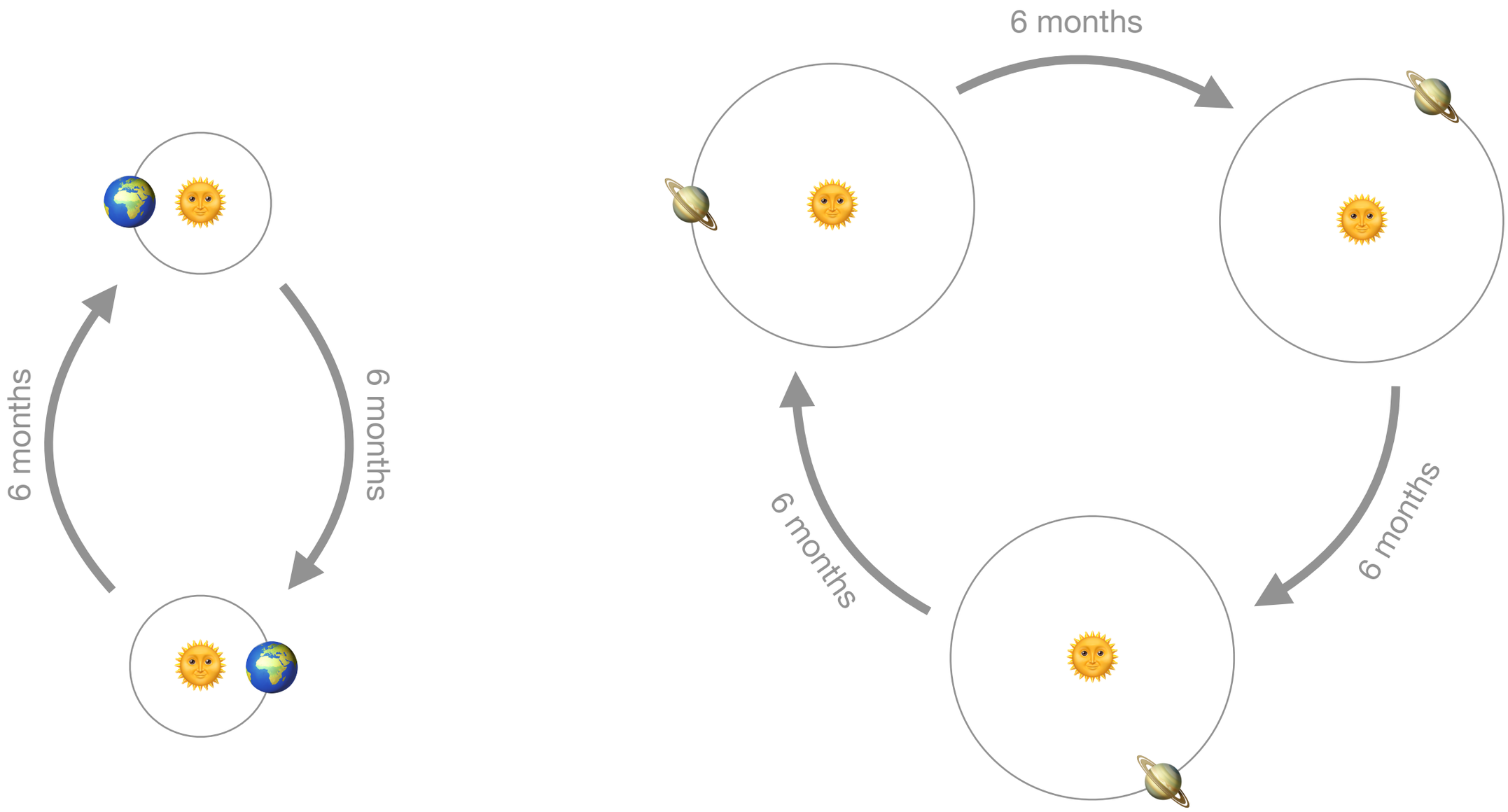

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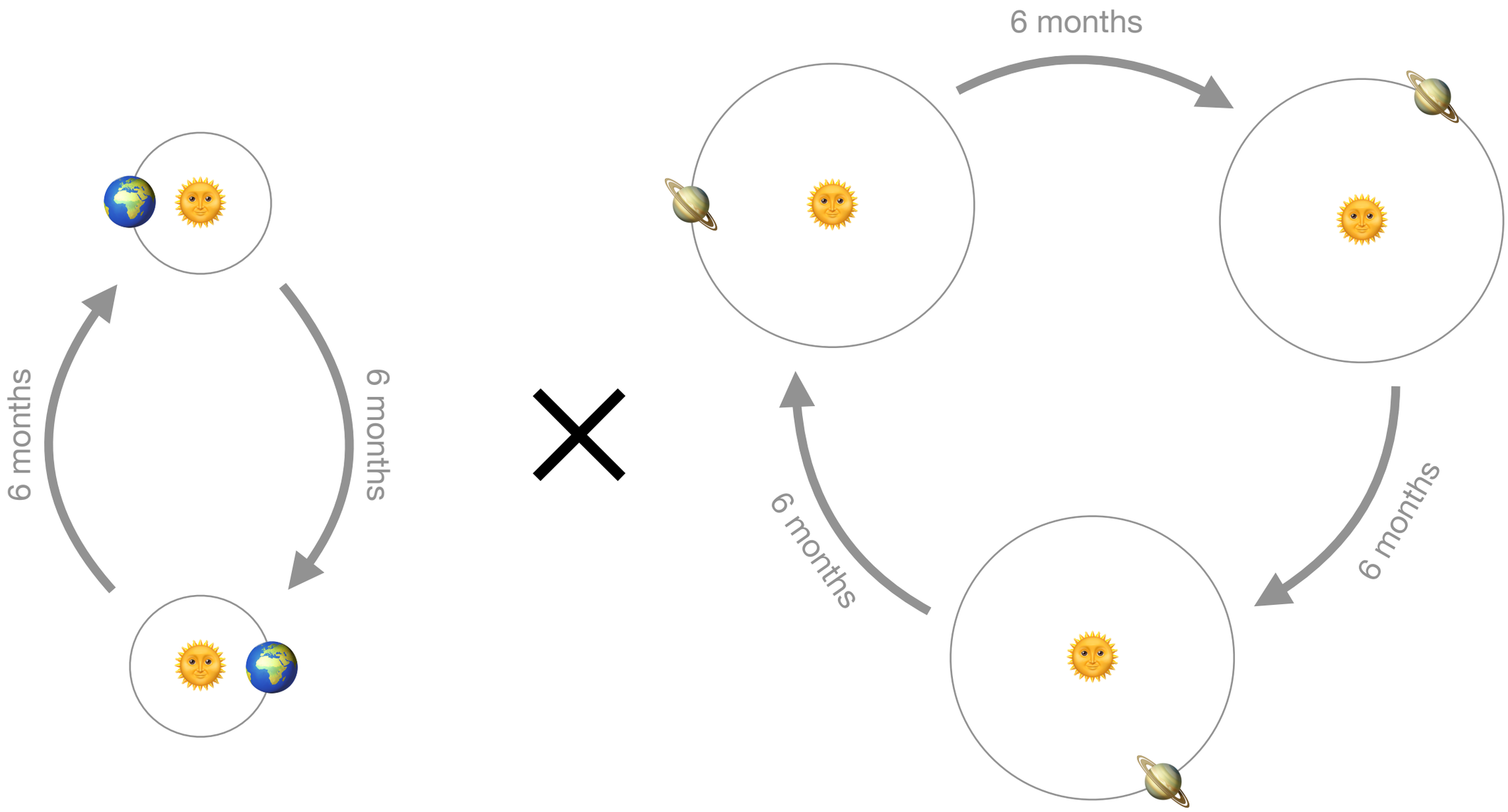
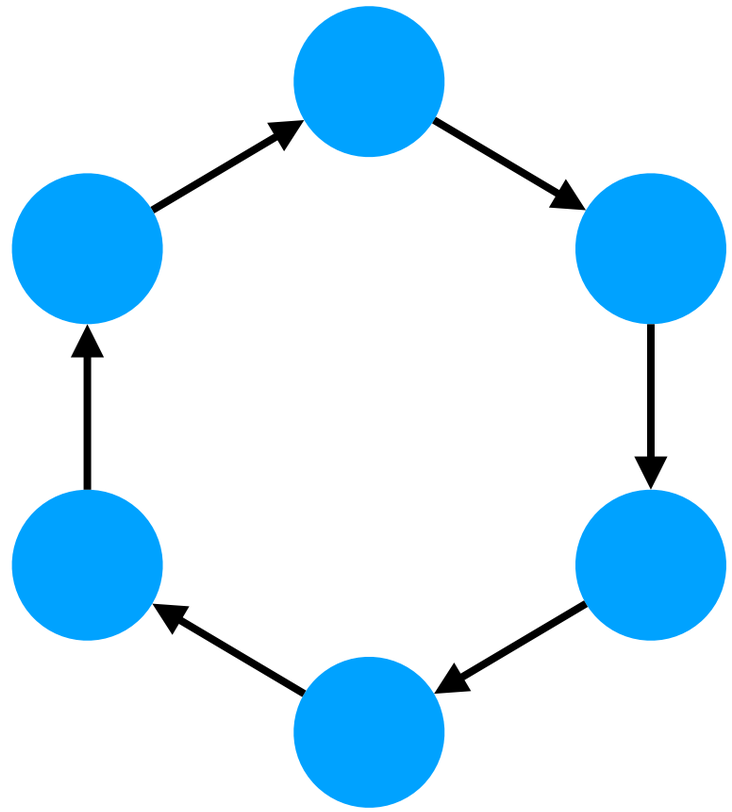


What if our instruments are less sophisticated?

Abstract evolution of the system



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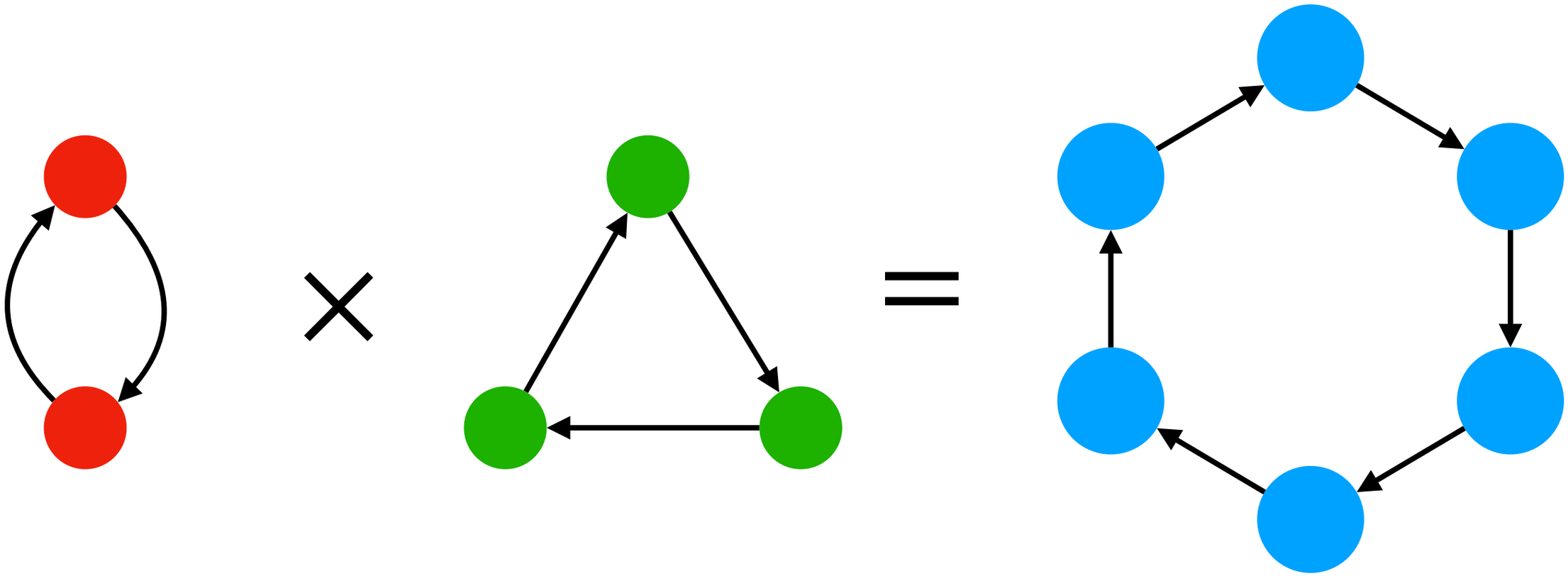

Product of dynamical systems

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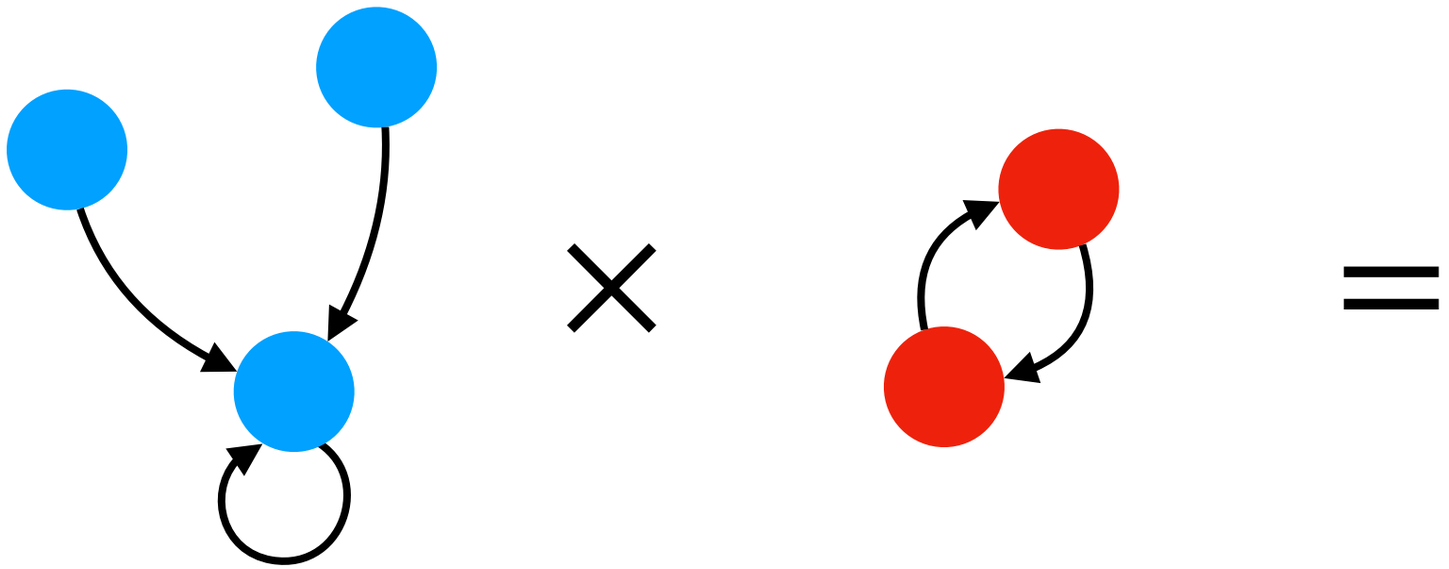
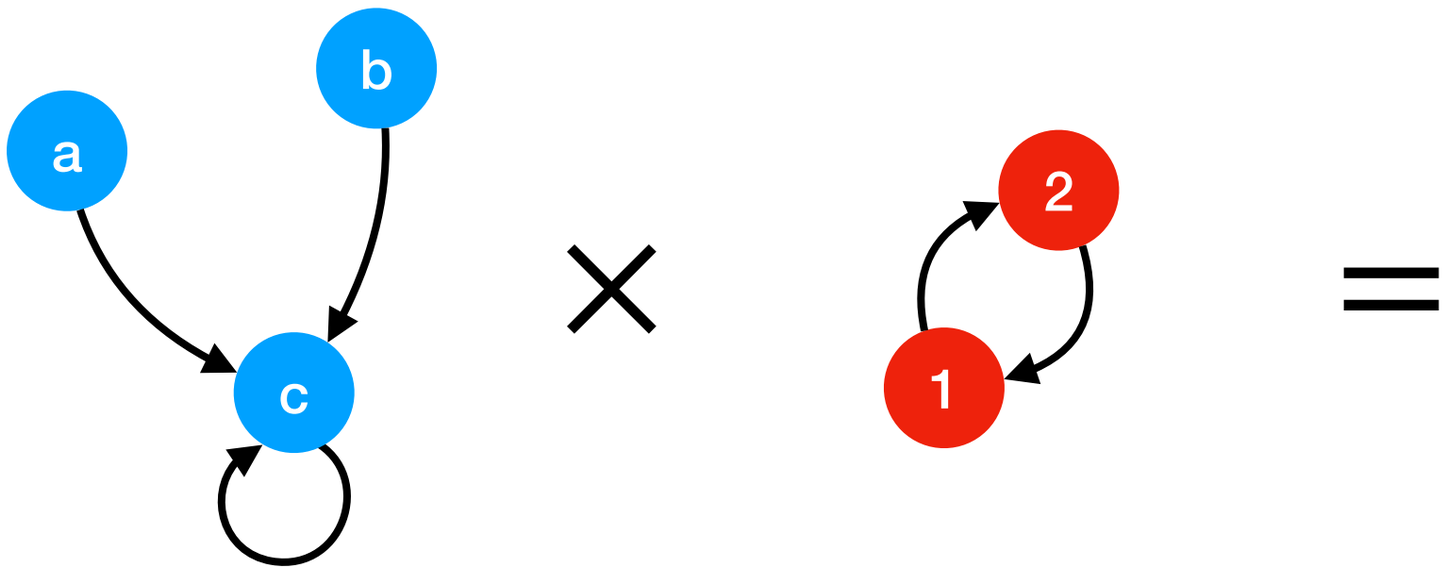
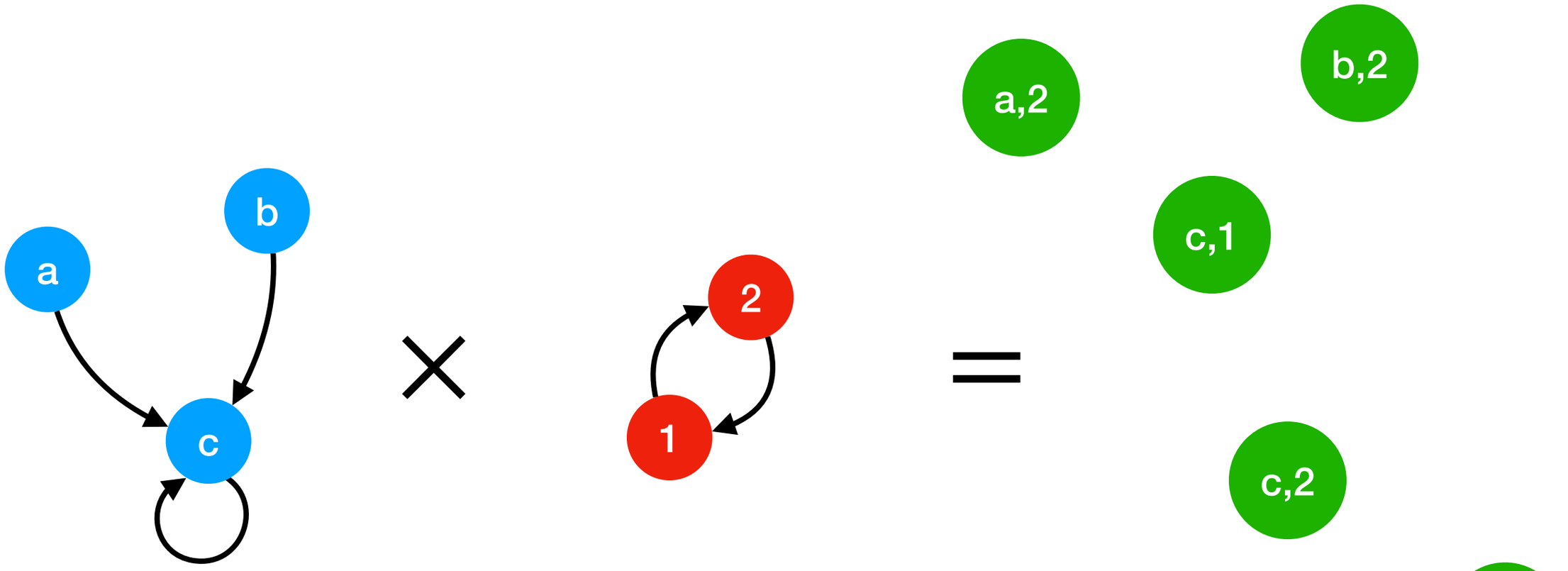


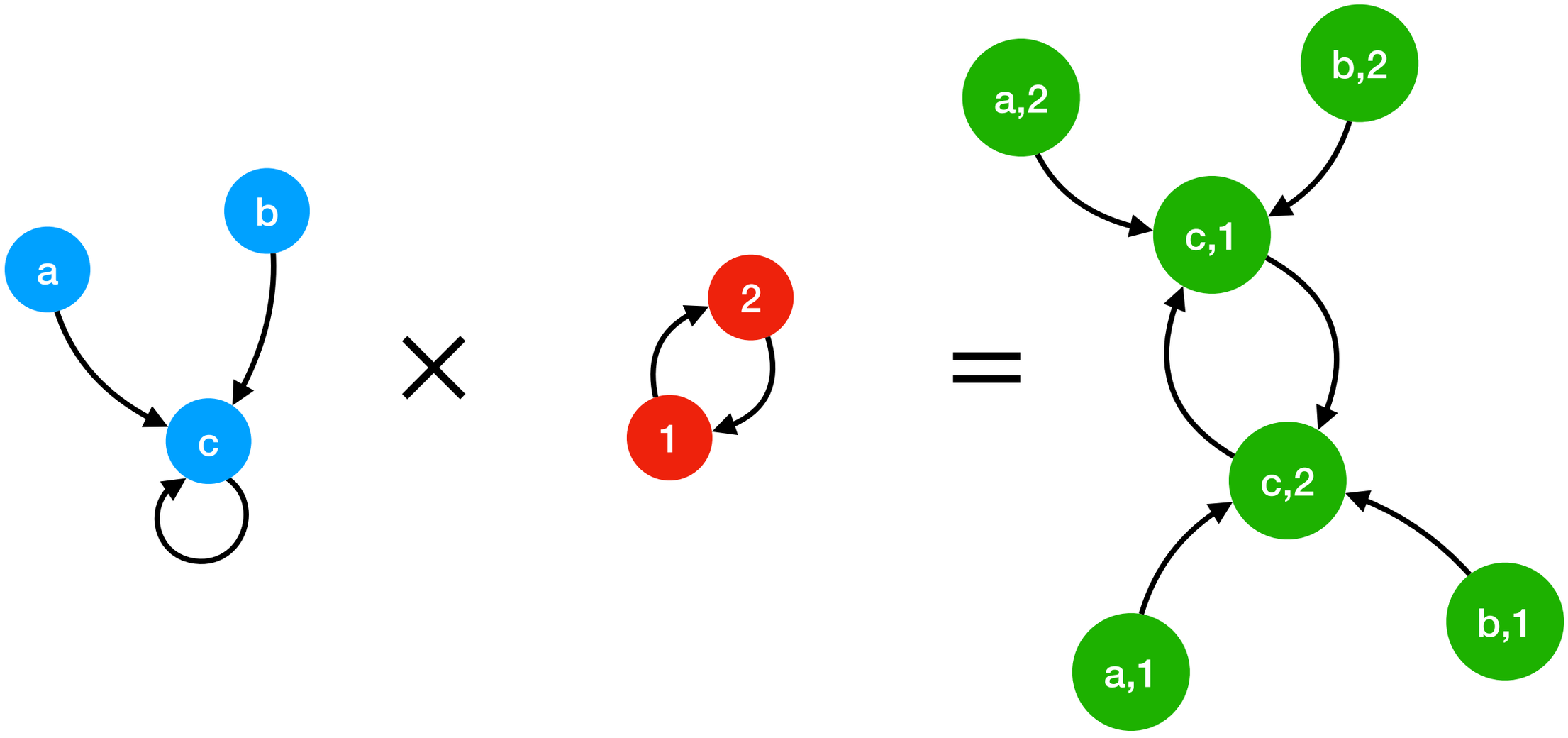
Give temporary names to the states



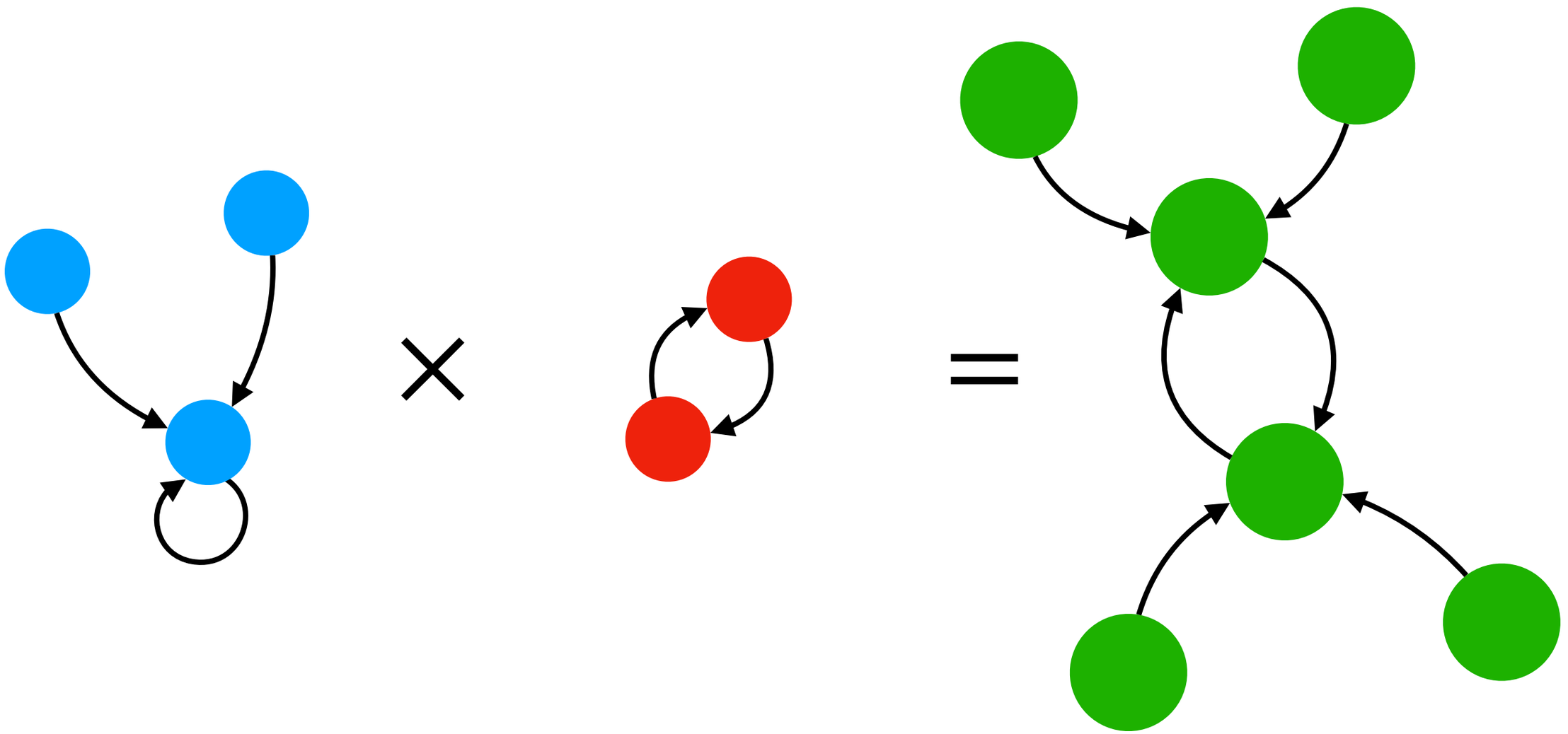
Compute the Cartesian product



Add the arcs between states

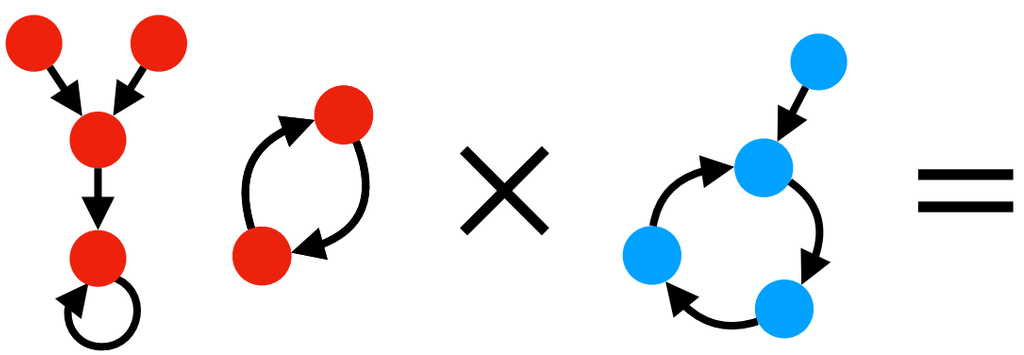


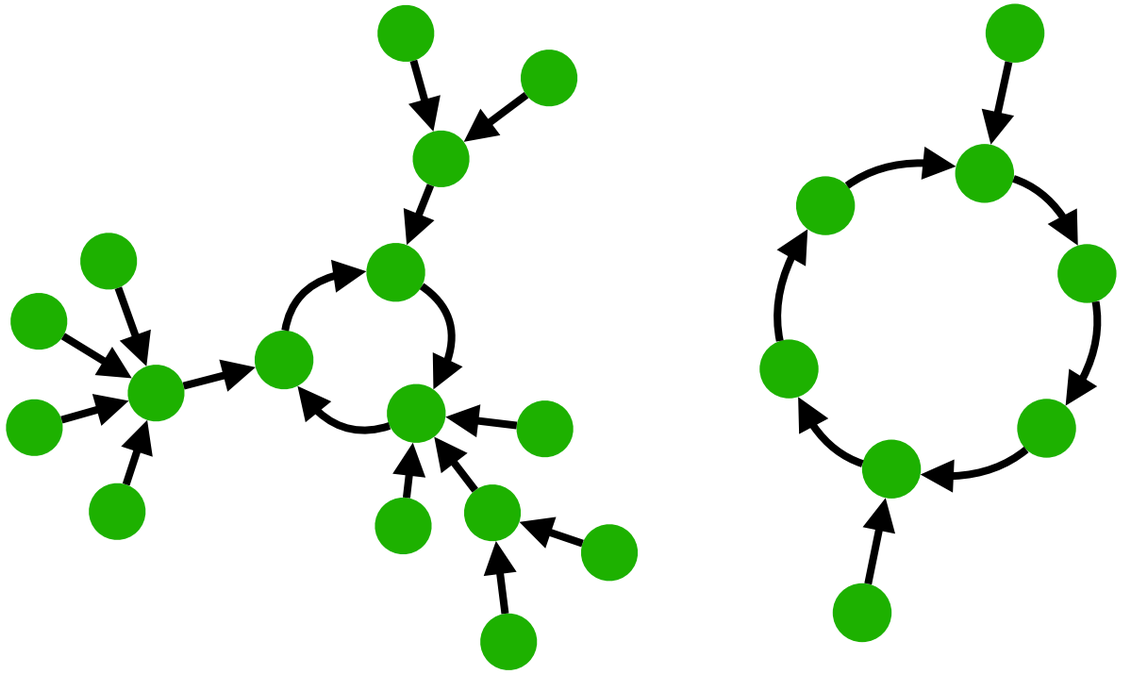
Forget the names once again



Products “preserve” behaviours

A is a minor of $A \times B$ for $B \neq \emptyset$





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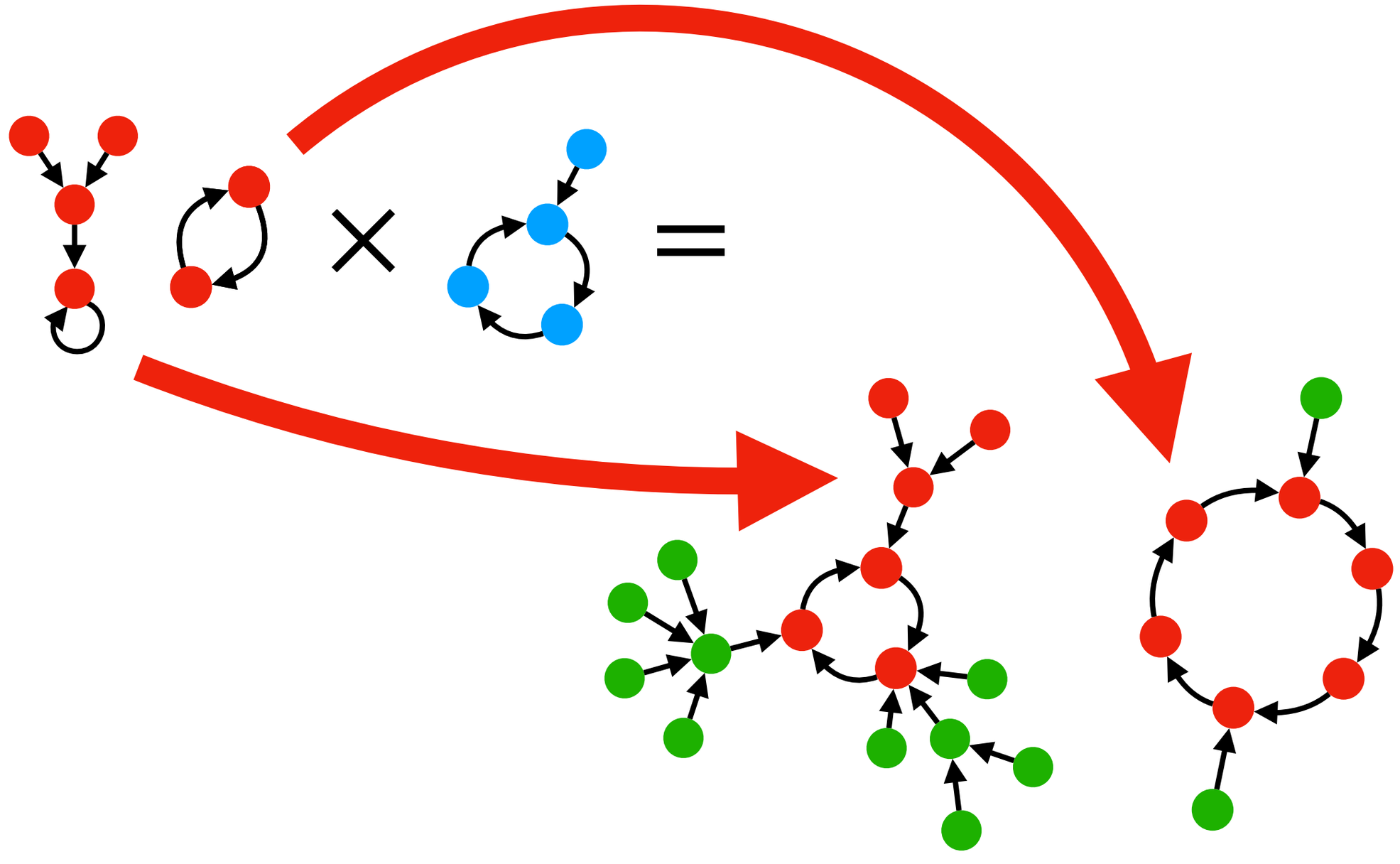

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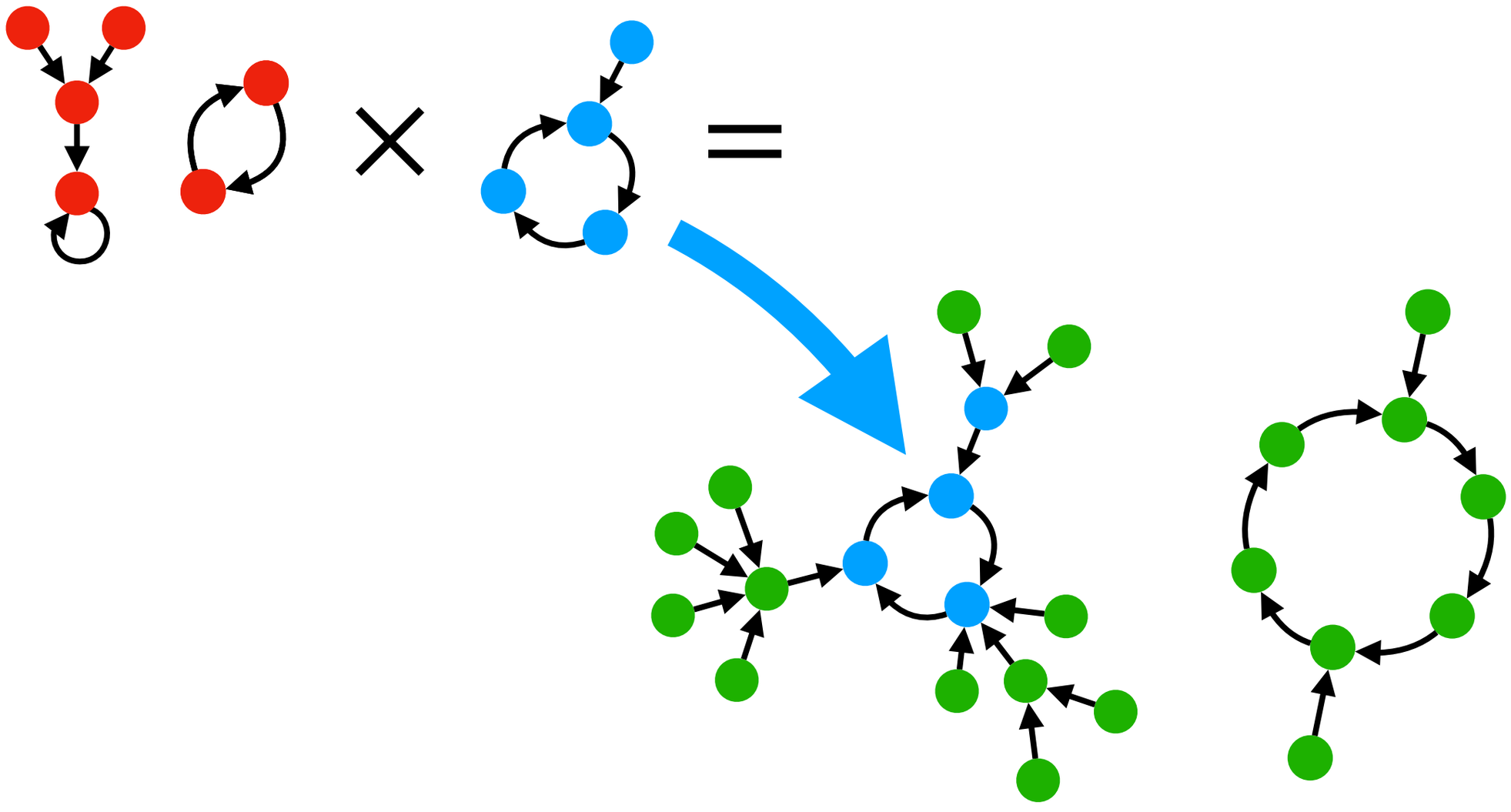

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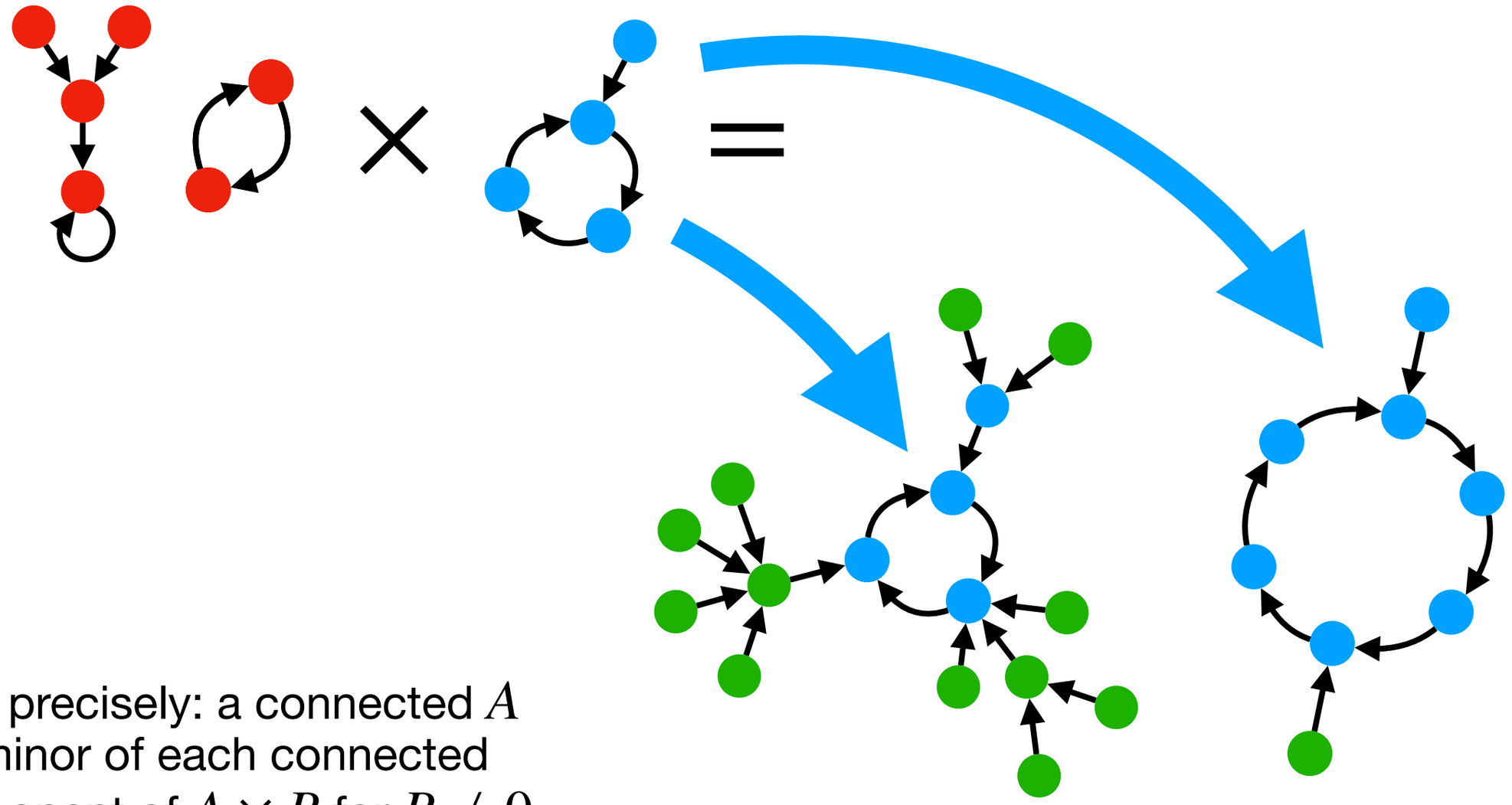
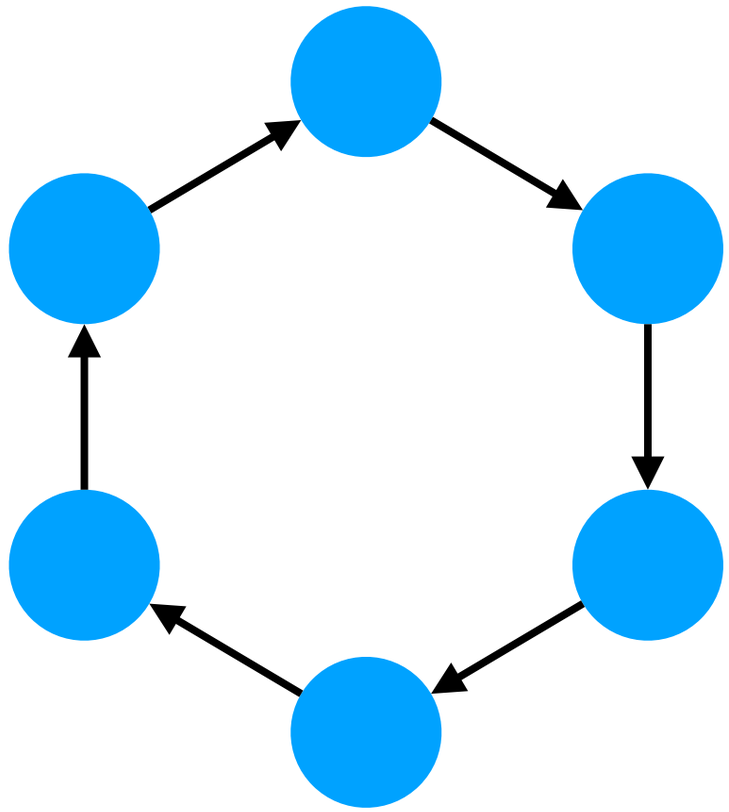


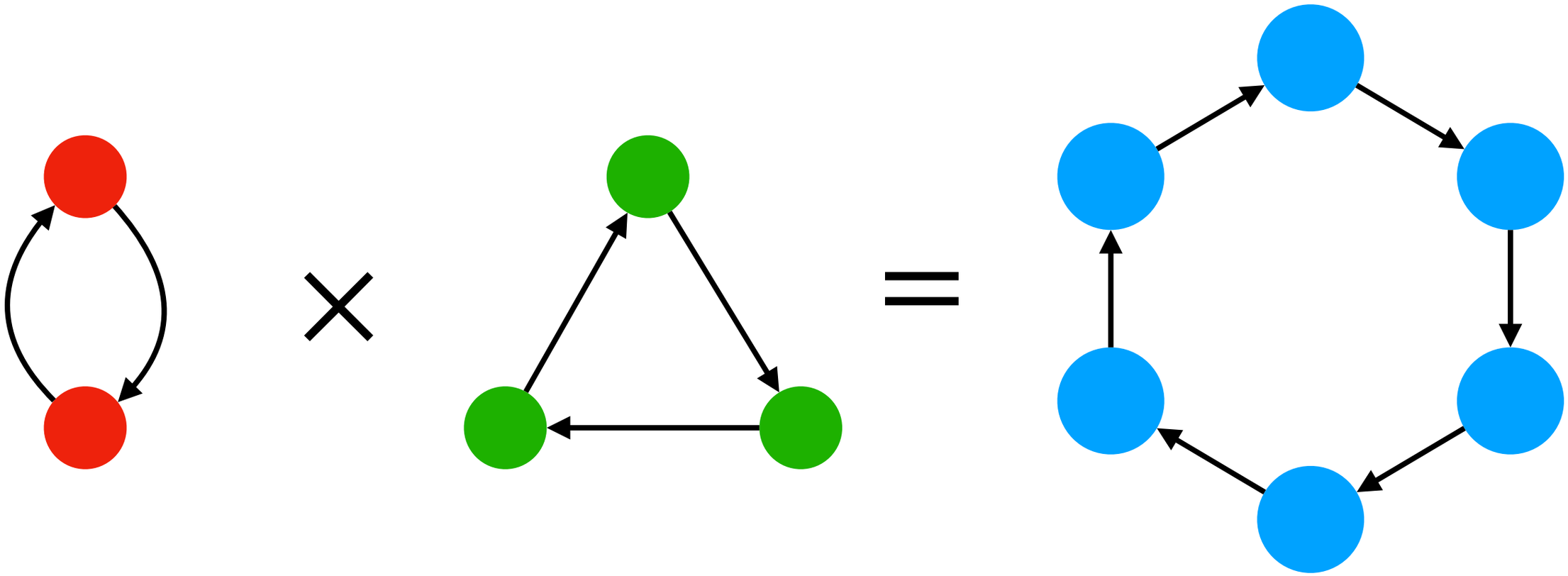
more precisely: a connected A is a minor of each connected component of $A \times B$ for $B \neq 0$

Back to our planetary system

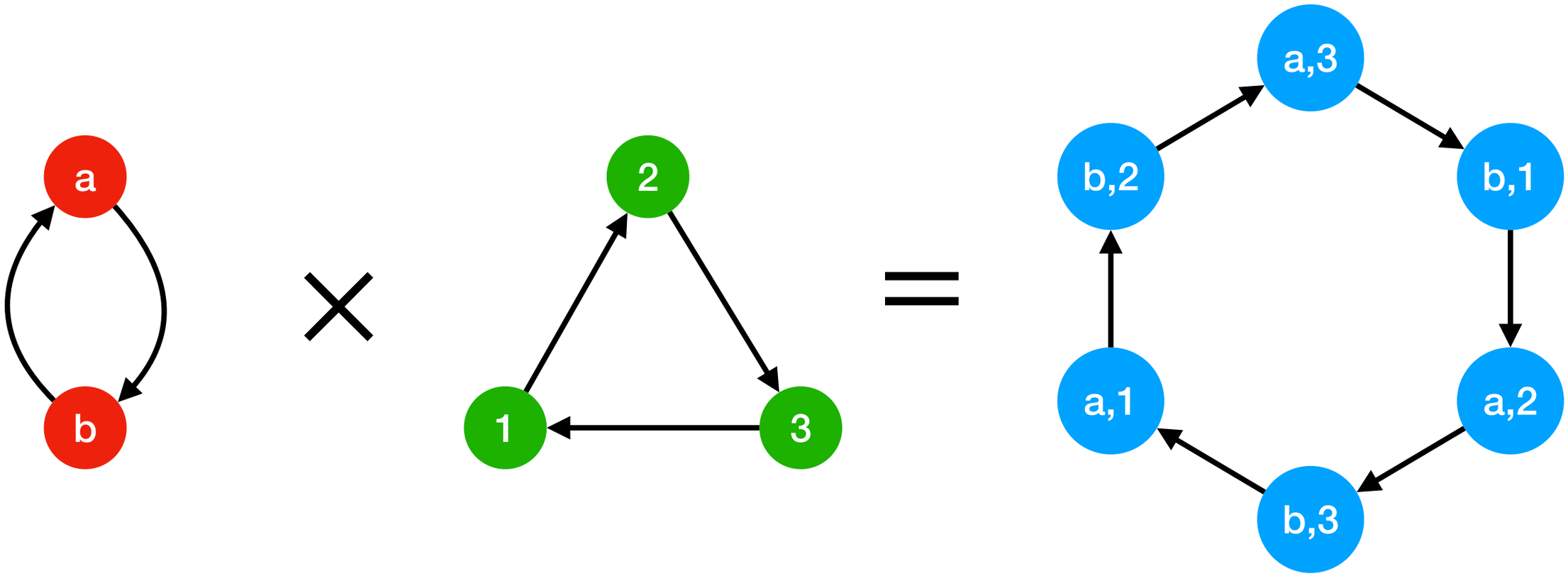
Decomposition



Decomposition



Decomposition



Any other decomposition?

Another decomposition

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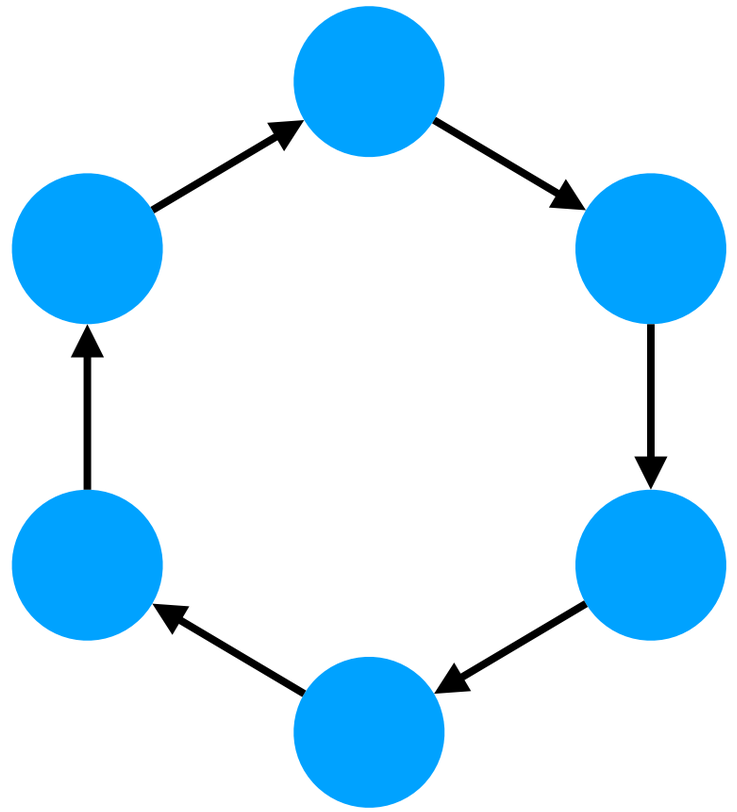
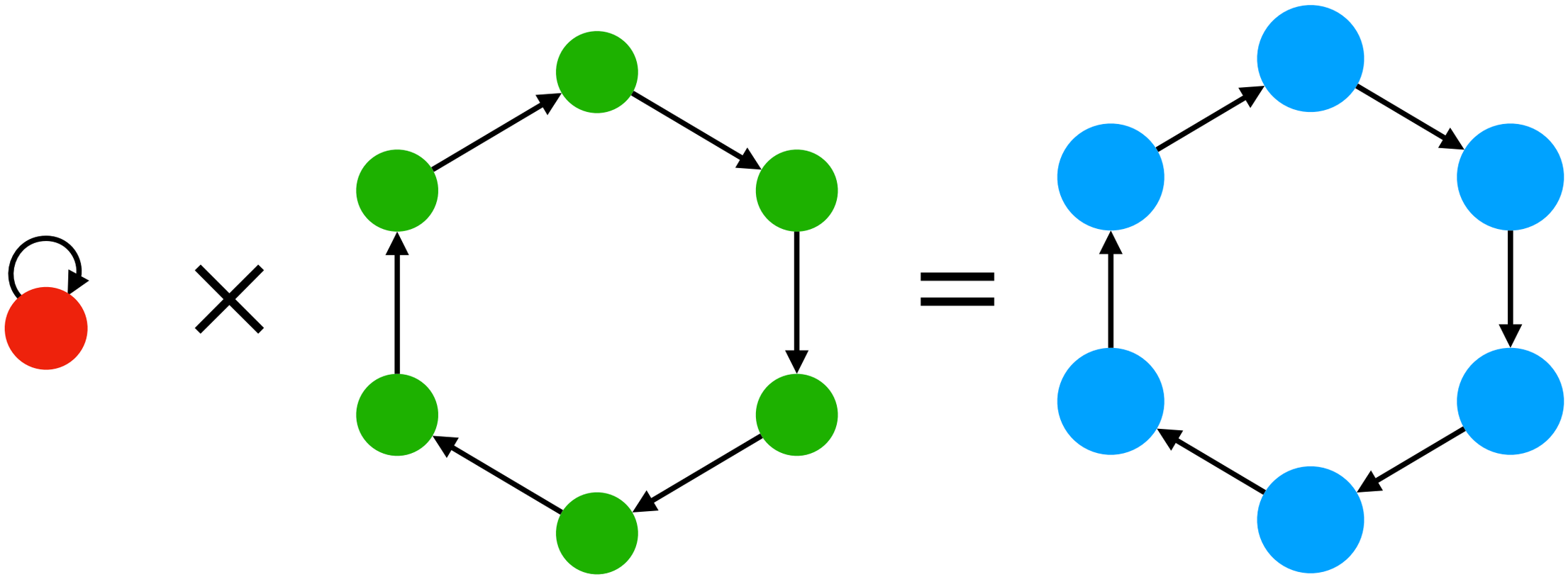


Another decomposition



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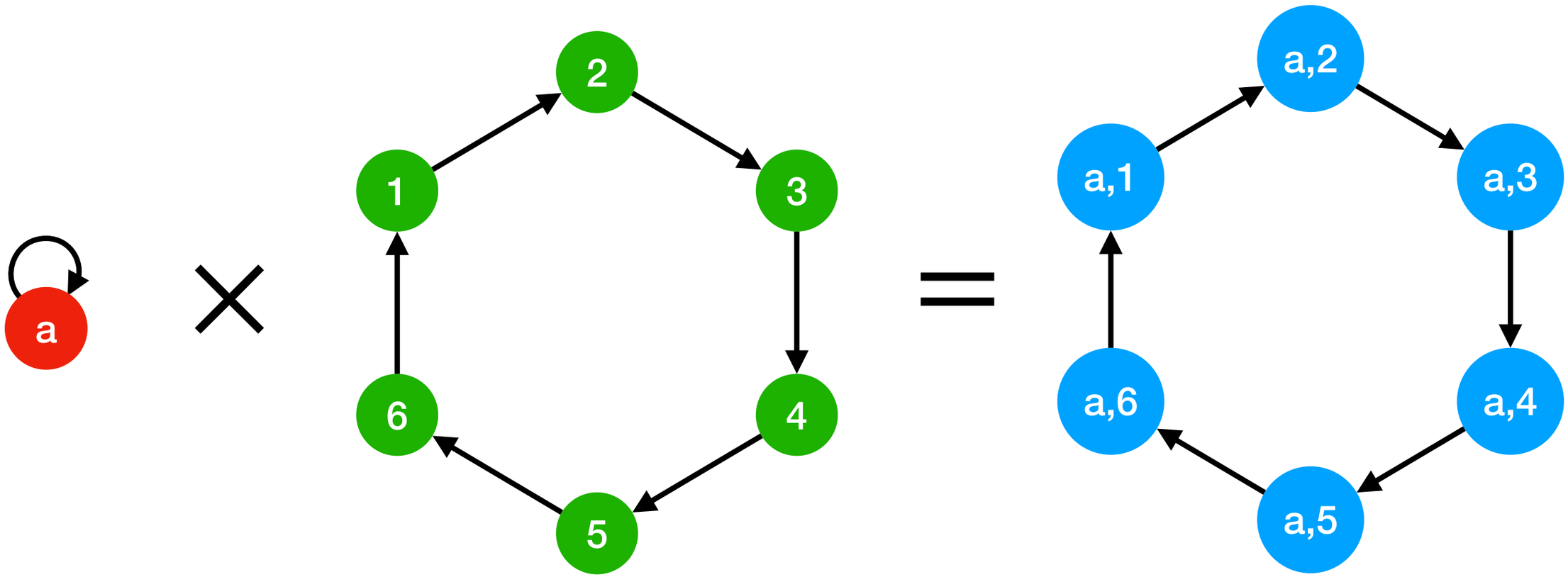

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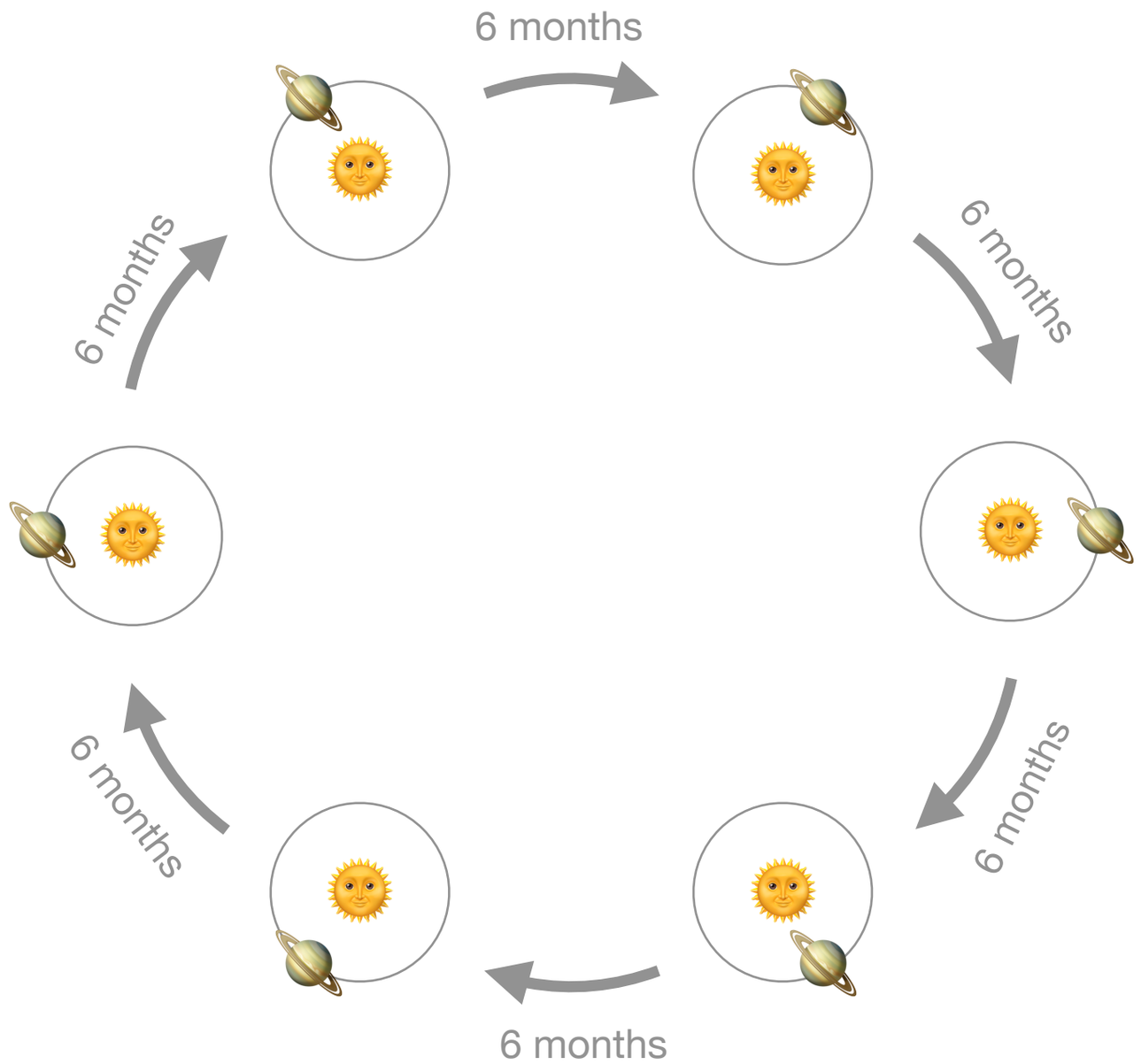

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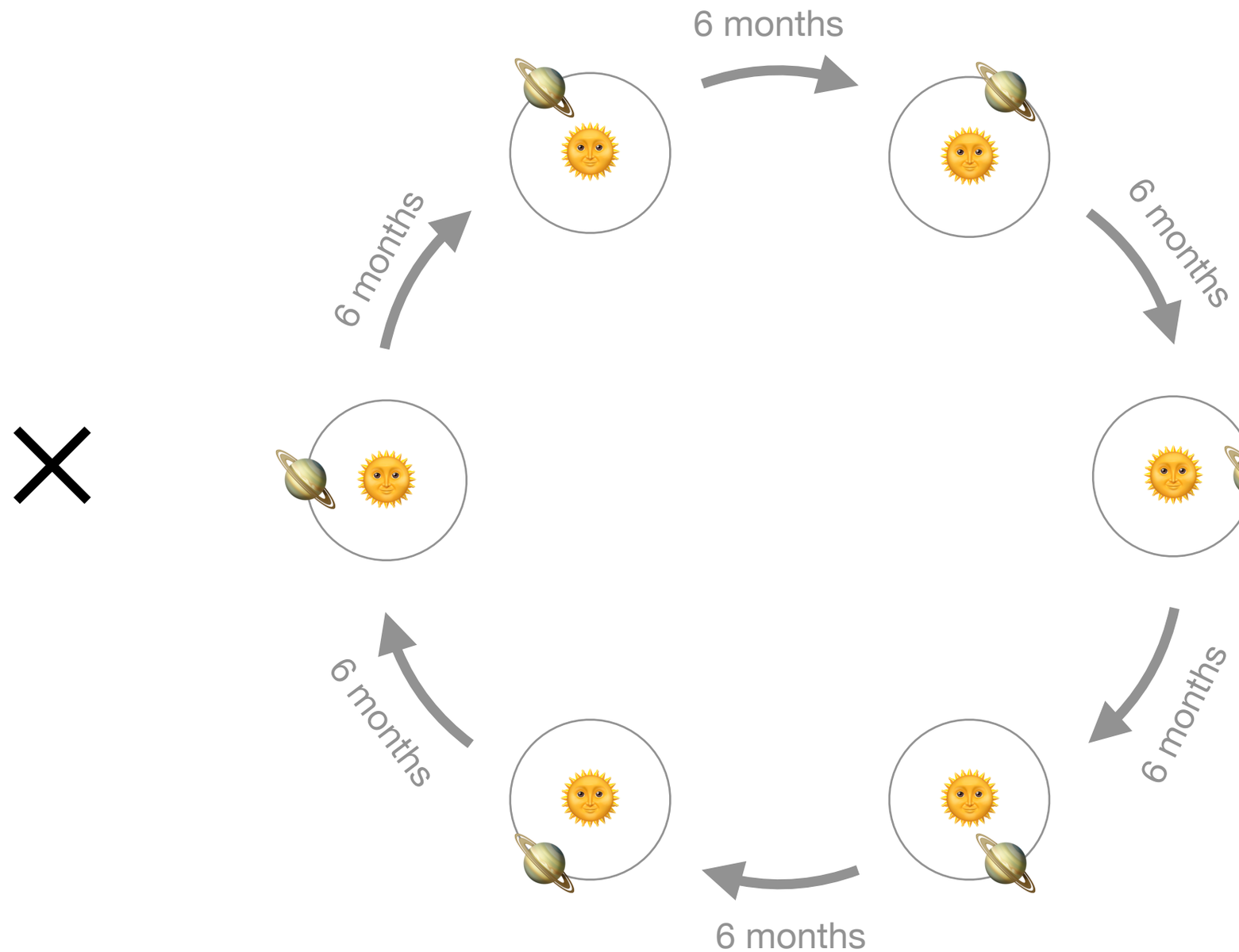

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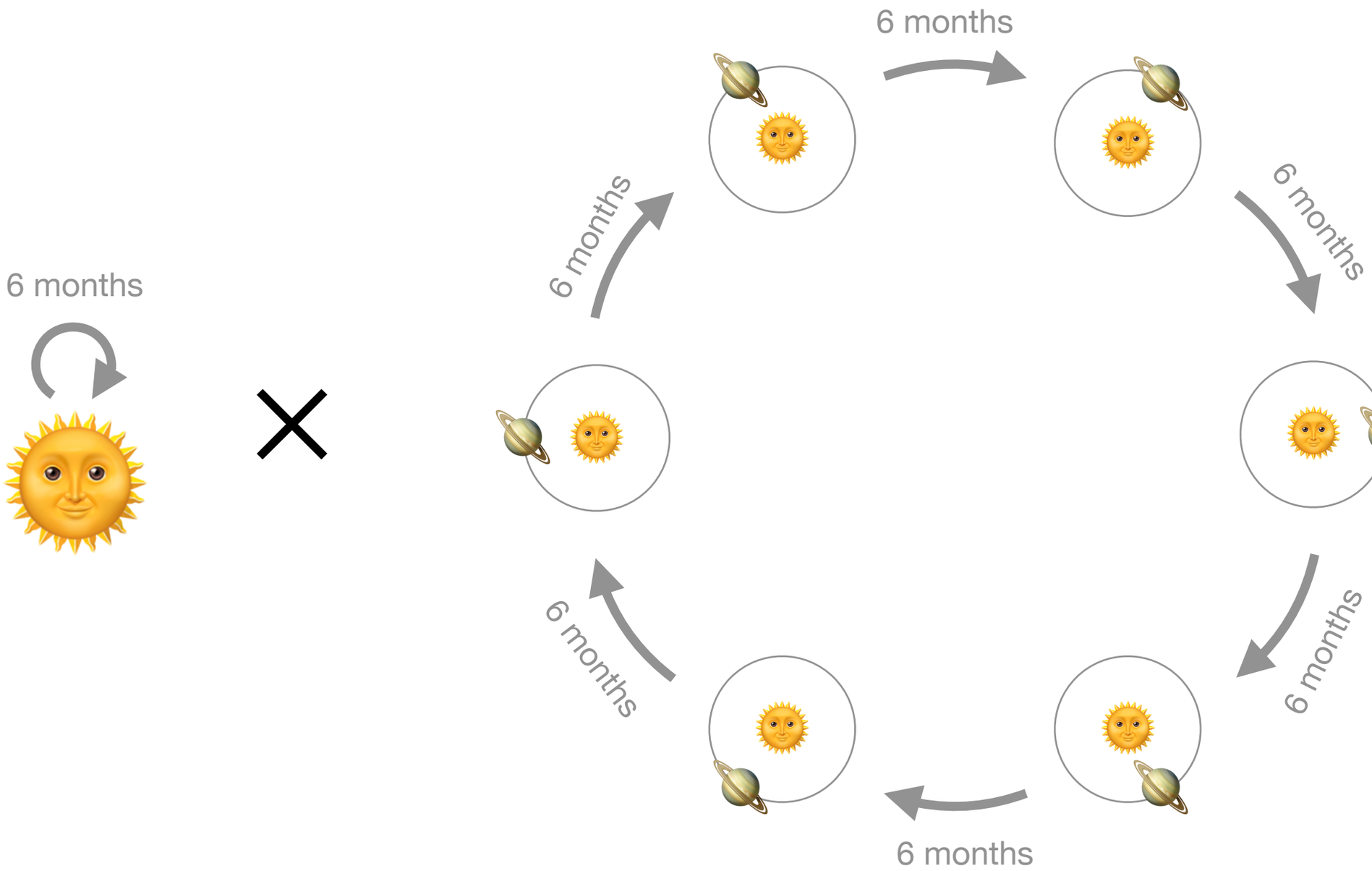
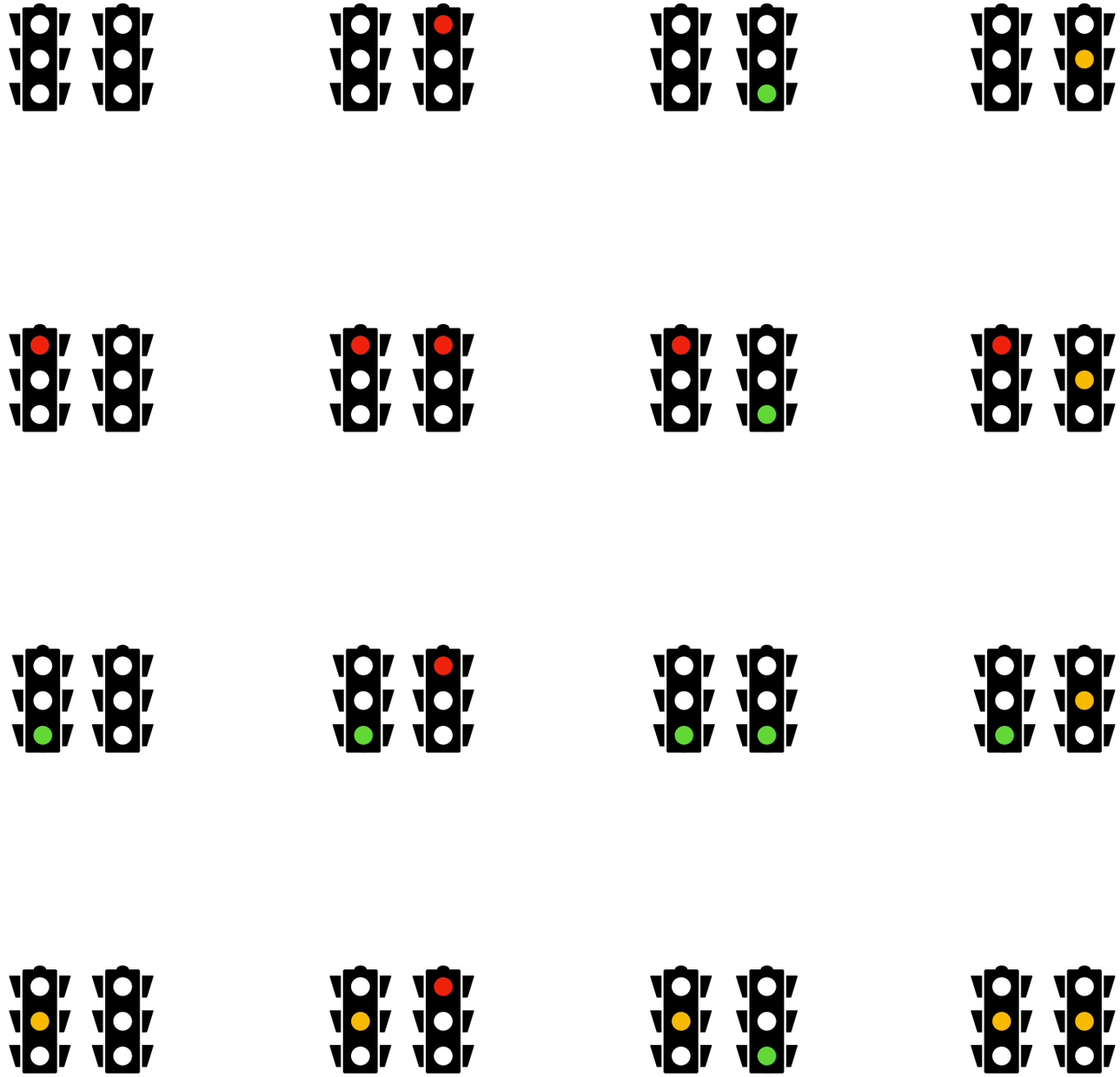


Untangling complex systems

Traffic lights at a crossroads



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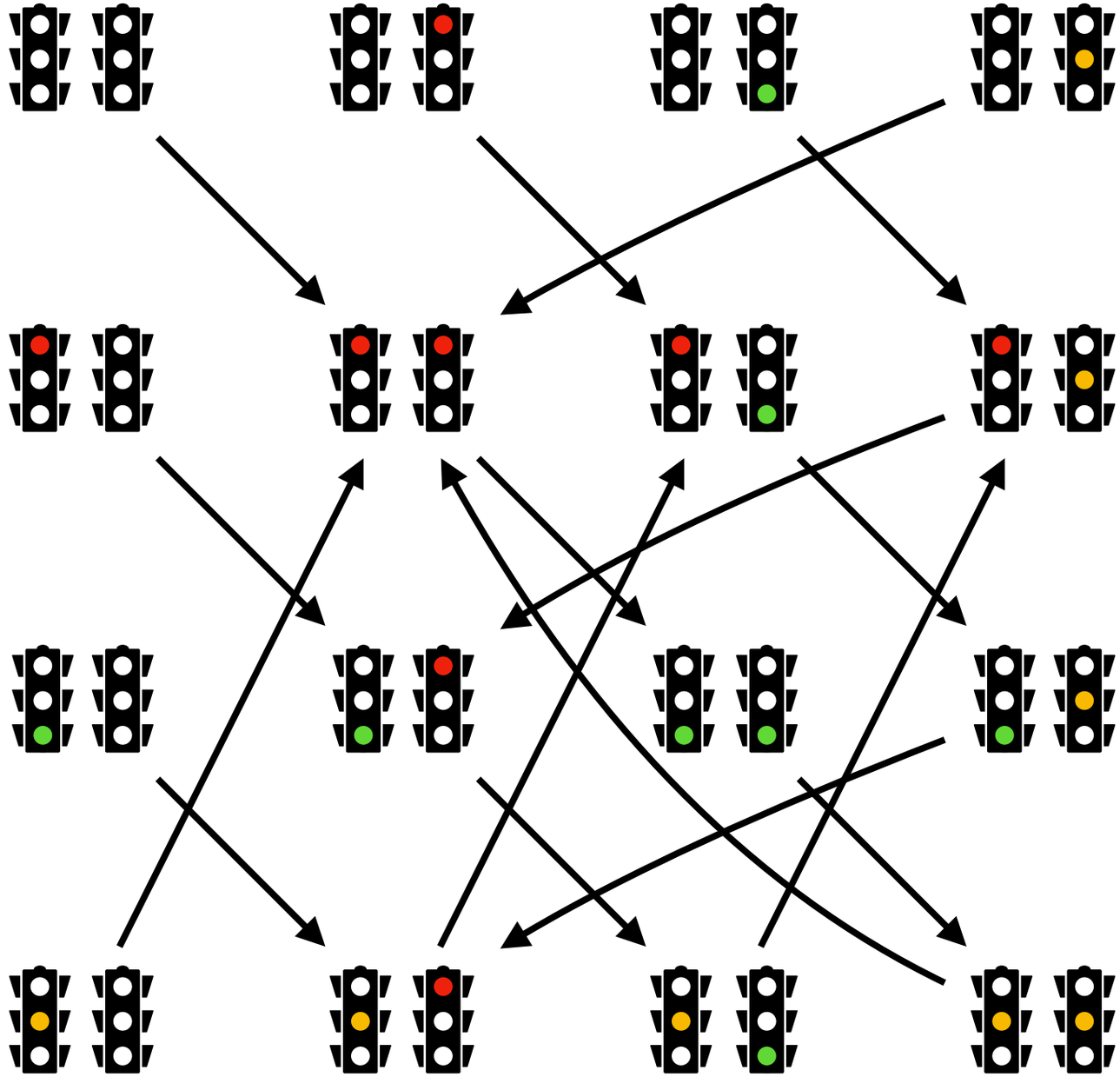

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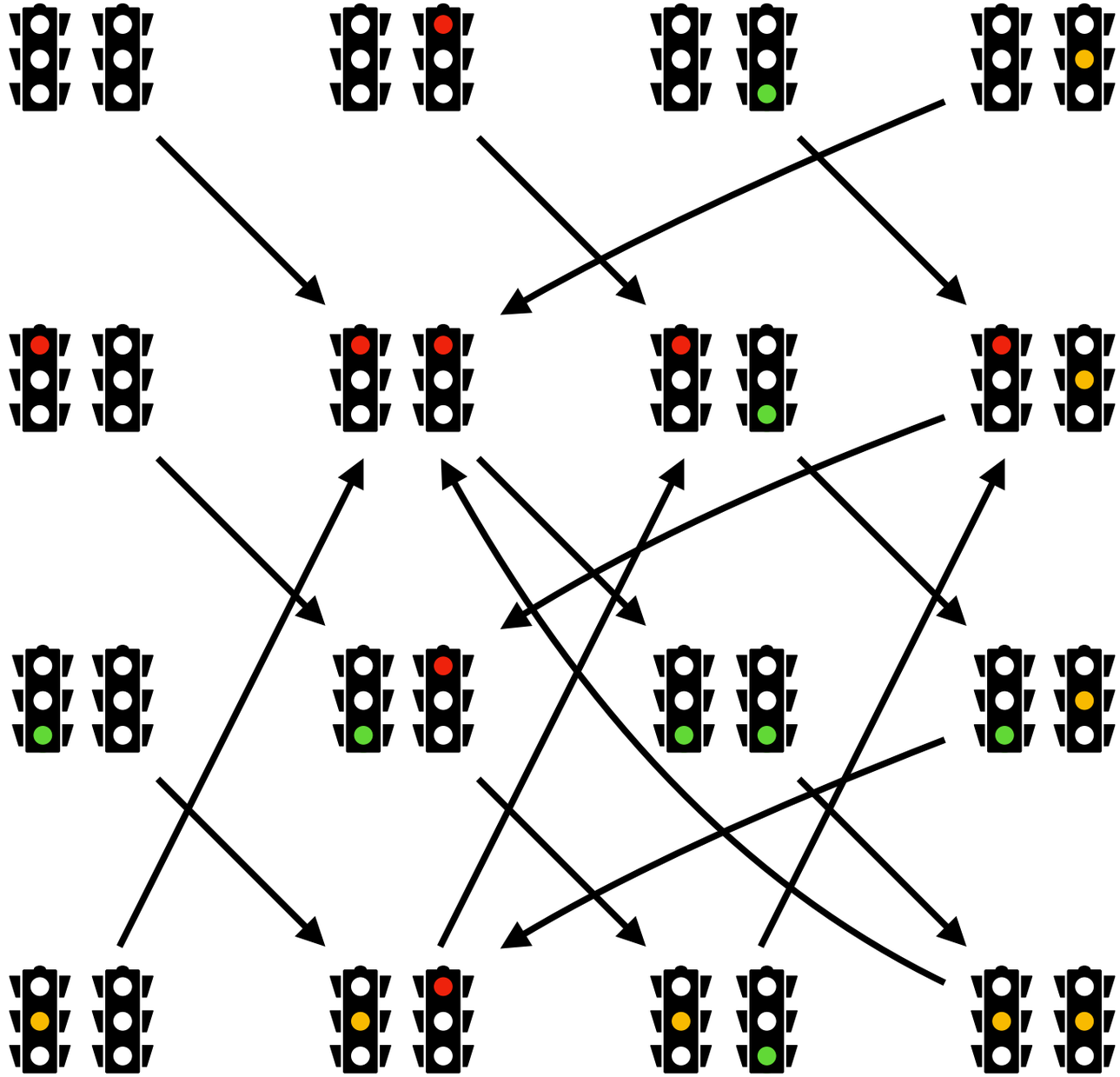

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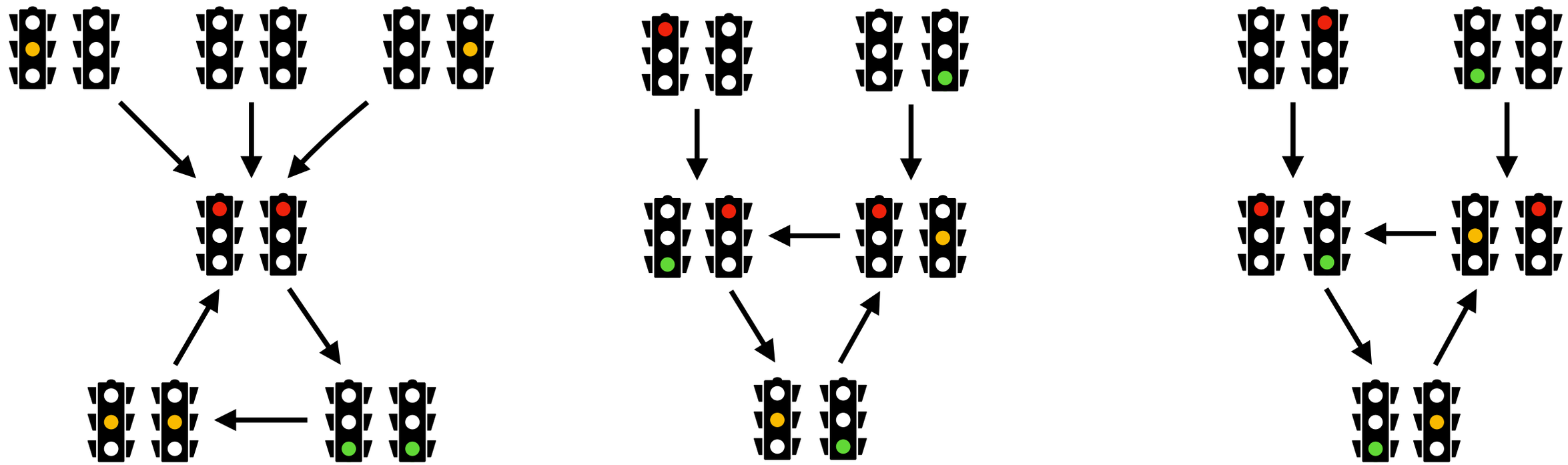

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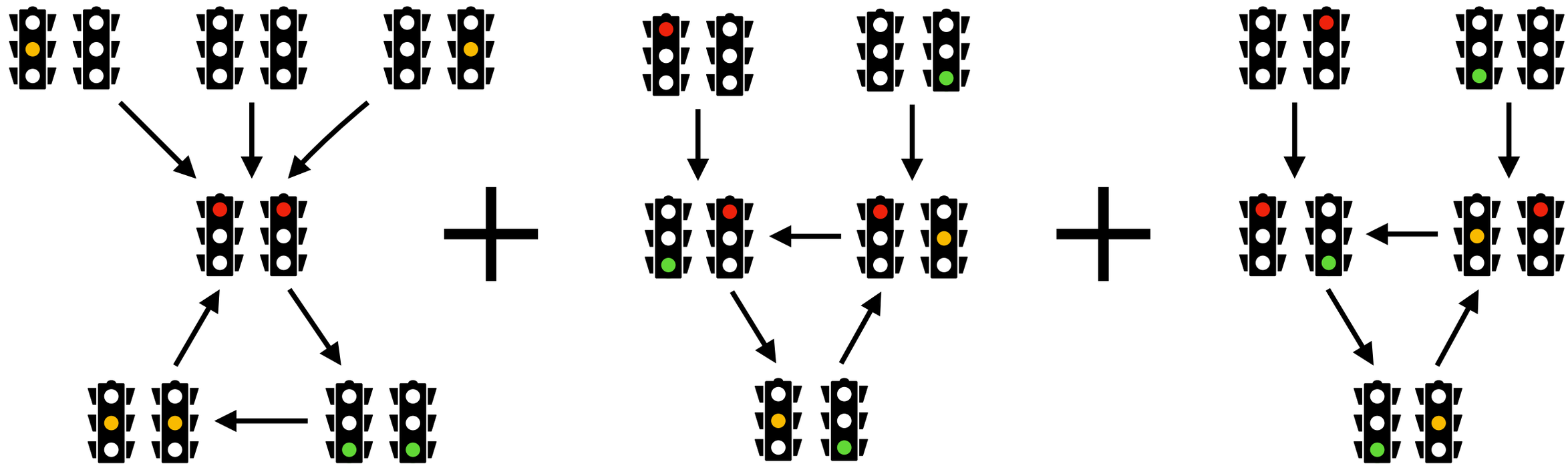
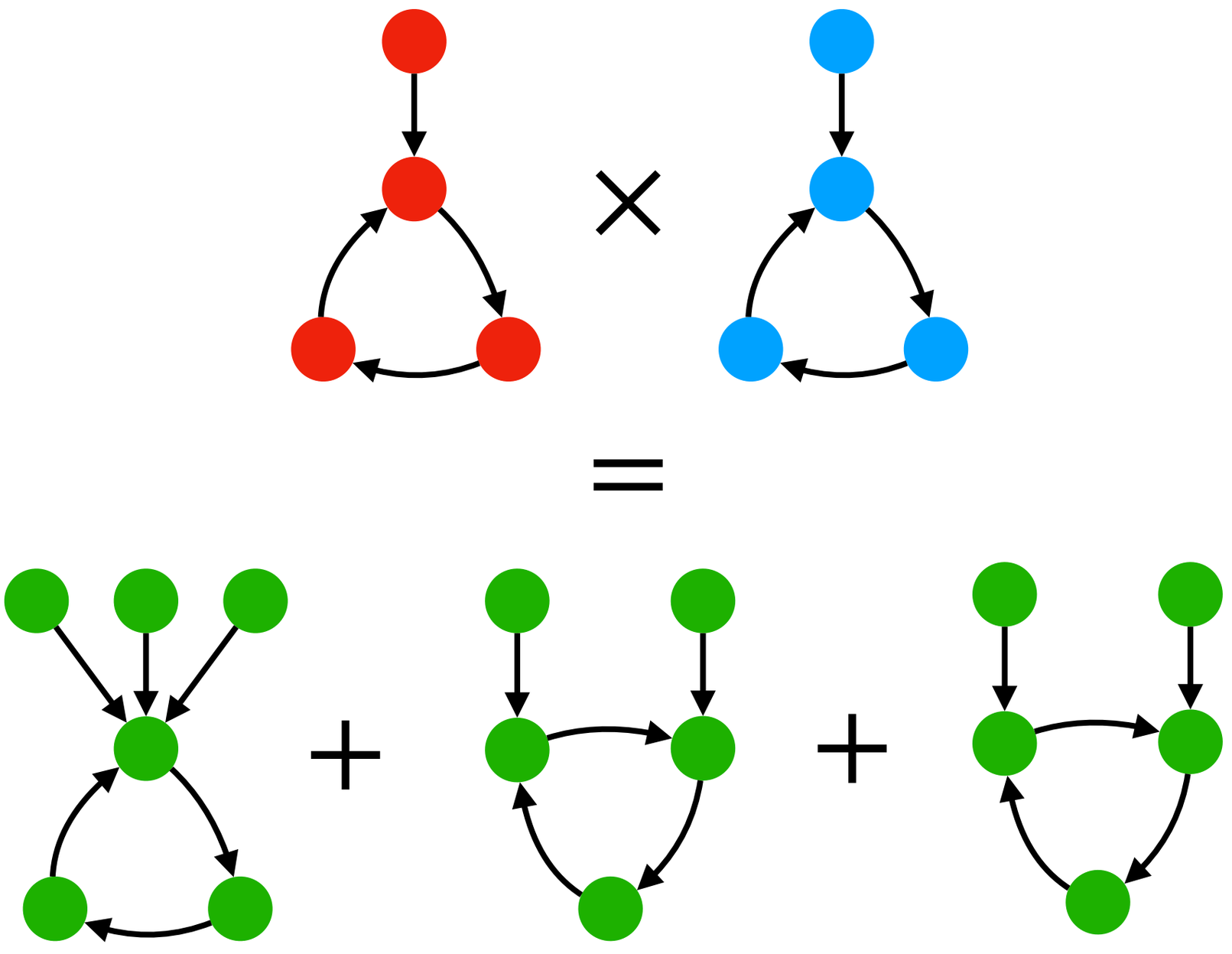


More abstractly…



The operations $+$ and $\times$ are a commutative semiring

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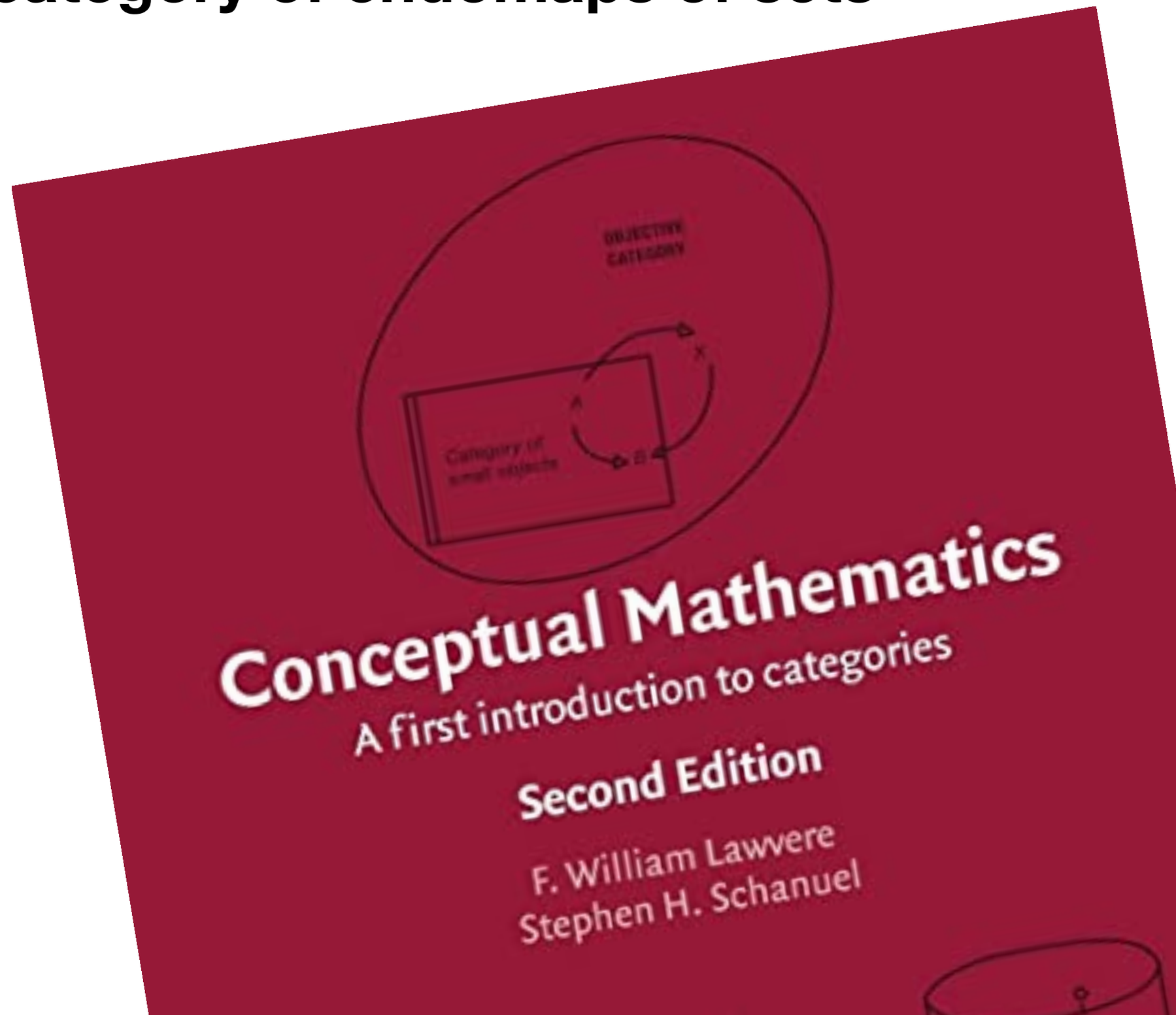
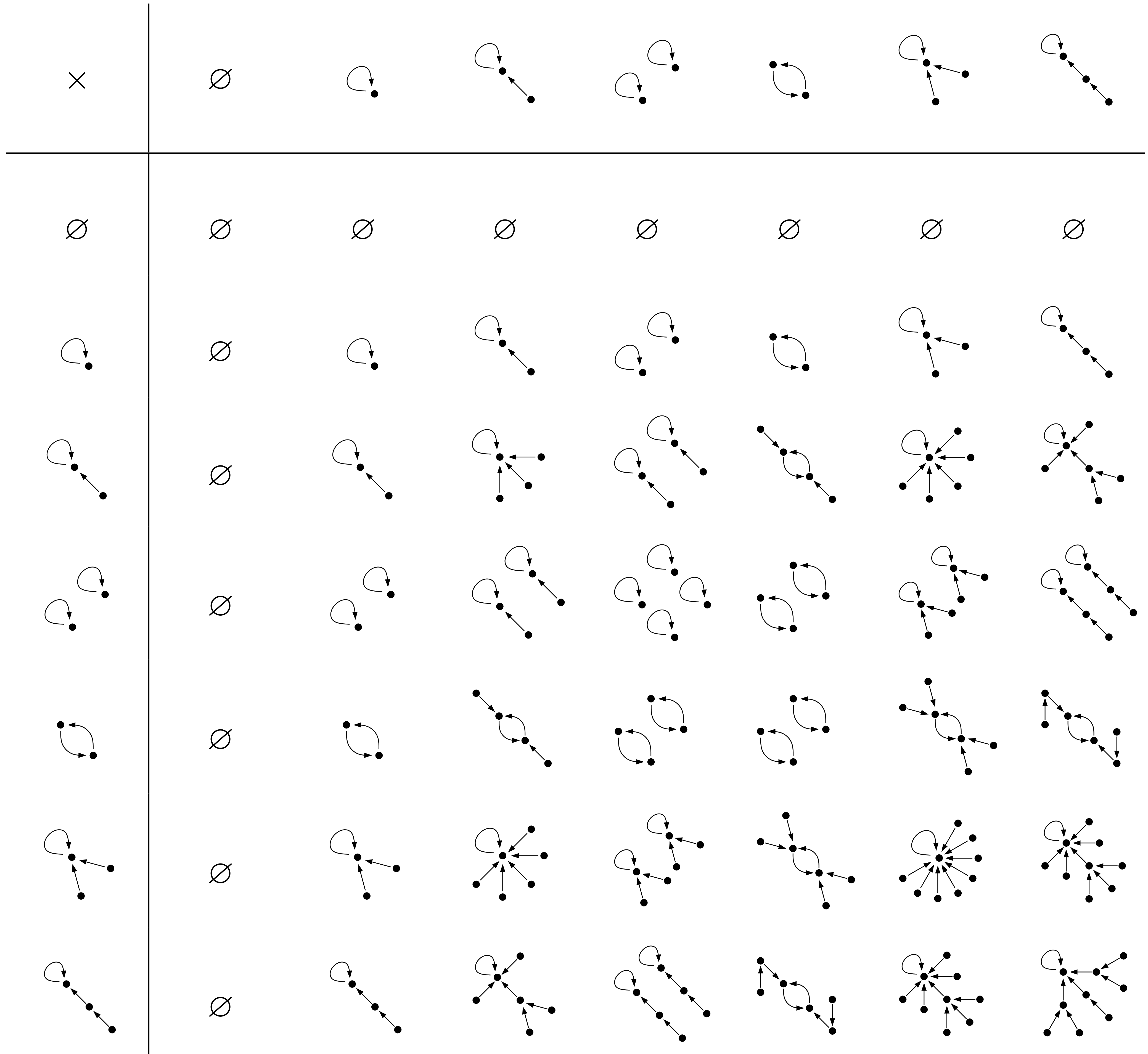
- Neutral elements: $\varnothing + X = X$ and $\text{🎧} \times X = X$
- Distributive: $X \times (Y + Z) = X \times Y + X \times Z$
- Multiplication by zero: $\varnothing \times X = \varnothing$

The inspiration

The category of endomaps of sets

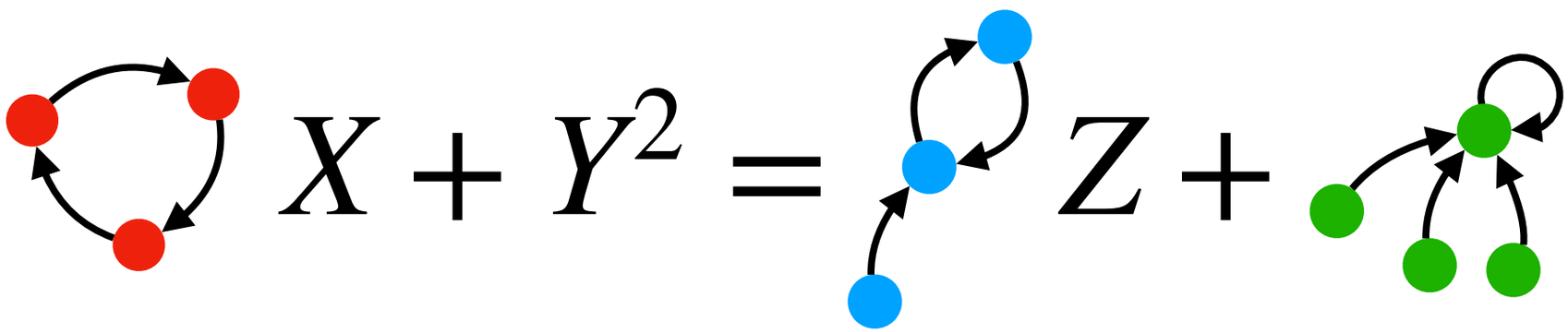


Multiplication table



Equations for decomposing systems

Eqns over dynamical systems



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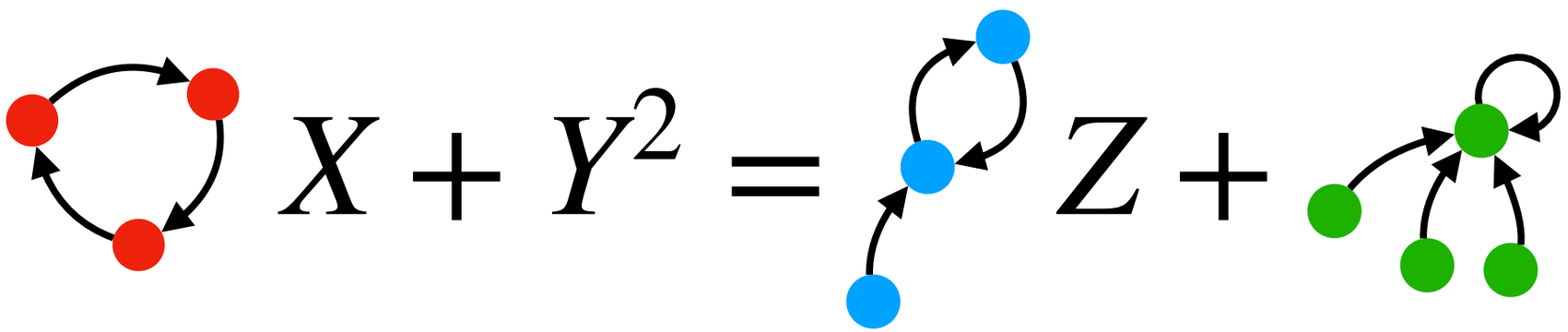
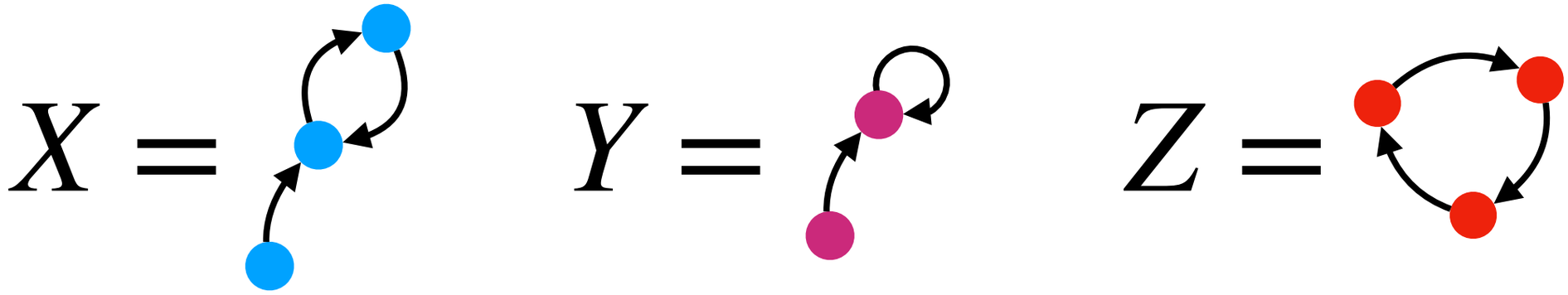




$\mathbb{N}$ is a subsemiring of $\mathbf{D}$

- There is an injective homomorphism $\varphi : \mathbb{N} \to \mathbf{D}$

$$\varphi(n) = \mathbf{1} + \mathbf{1} + \cdots + \mathbf{1} = \underbrace{\quad + \quad + \cdots +}_{n \text{ times}} \qquad \underbrace{\quad}_{n \text{ times}}$$

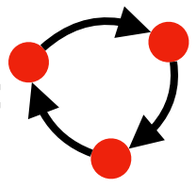
- n fixed points behave exactly as the integer n

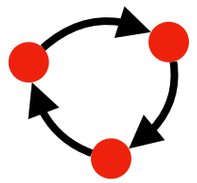
- So $\mathbf{D}$ contains a isomorphic copy of $\mathbb{N}$

Natural polynomial equations

- Let $p(X, Y) = 2X^2$ and $q(X, Y) = 3Y$ with $p, q \in \mathbb{N}[X, Y] \leq \mathbf{D}[X, Y]$

- Then $2X^2 = 3Y$ has the non-natural solution

$X =$ 

$Y = 2$ 

- But, of course, it also has the natural solution $X' = 3$, $Y' = 6$

- Notice how $X' = |X|$ and $Y' = |Y|$

- This is not a coincidence!

The function "size" $|\cdot| : \mathbf{D} \to \mathbb{N}$

It's a semiring homomorphism

- $|\varnothing| = 0$
- $|$  $| = 1$

- Since $+$ is the disjoint union, we have

$$|A + B| = |A| + |B|$$

- Since $\times$ is the cartesian product, we have

$$|AB| = |A| \times |B|$$

Notation for polynomials $p \in \mathbf{D}[\overrightarrow{X}]$

Of degree $\leq d$ over the variables $\overrightarrow{X} = (X_1, \ldots, X_k)$

$$p = \sum_{\vec{i} \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{X}^{\vec{i}}$$

where $\overrightarrow{X}^{\vec{i}} = \prod_{j=1}^{k} X_j^{i_j}$

Theorem

Solvability of natural equations

- If a polynomial equation over $\mathbb{N}[X_1, \ldots, X_k]$ has a solution in $\mathbf{D}^k$, then it also has a solution in $\mathbb{N}^k$

- In the larger semiring $\mathbf{D}$ we may find extra solutions, but only if the equation is already solvable over the naturals

- Then, by reduction from Hilbert's 10th problem, we obtain the undecidability in $\mathbf{D}$ of equations over $\mathbb{N}[\vec{X}]$…

- …and thus of arbitrary equations over $\mathbf{D}[\vec{X}]$

Proof

Consider $p(\overrightarrow{X}) = q(\overrightarrow{X})$ **with** $p, q \in \mathbb{N}[\overrightarrow{X}]$

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{X}^{\vec{i}} = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \overrightarrow{X}^{\vec{i}}$$

Proof

Suppose that $\overrightarrow{A} \in \mathbf{D}^k$ is a solution

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{A^{\vec{i}}} = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \overrightarrow{A^{\vec{i}}}$$

Proof

Apply the size function $|\cdot|$

$$\left| \sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{A^{\vec{i}}} \right| = \left| \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \overrightarrow{A^{\vec{i}}} \right|$$

Proof

The size function $|\cdot|$ is a homomorphism

$$\sum_{i \in \{0,\ldots,d\}^k} \left| a_{\vec{i}} \overrightarrow{A^{\vec{i}}} \right| = \sum_{i \in \{0,\ldots,d\}^k} \left| b_{\vec{i}} \overrightarrow{A^{\vec{i}}} \right|$$

Proof

The size function $|\cdot|$ is a homomorphism

$$\sum_{i \in \{0,\ldots,d\}^k} |a_{\vec{i}}| \left| \overrightarrow{A^{\vec{i}}} \right| = \sum_{i \in \{0,\ldots,d\}^k} |b_{\vec{i}}| \left| \overrightarrow{A^{\vec{i}}} \right|$$

Proof

The coefficients are natural

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \left| \overrightarrow{A^{\vec{i}}} \right| = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \left| \overrightarrow{A^{\vec{i}}} \right|$$

Proof

We have $\overrightarrow{A^{\vec{i}}} = \prod_{j=1}^{k} A_j^{i_j}$

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \left| \prod_{j=1}^{k} A_j^{i_j} \right| = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \left| \prod_{j=1}^{k} A_j^{i_j} \right|$$

Proof

The size function $|\cdot|$ is a homomorphism

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \prod_{j=1}^{} |A_j^{i_j}| = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \prod_{j=1}^{} |A_j^{i_j}|$$

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The size function $|\cdot|$ is a homomorphism

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Proof

So $|\overrightarrow{A}| = (|A_1|, \ldots, |A_k|)$ is also a solution, QED

$$p(|A_1|, \ldots, |A_k|) = q(|A_1|, \ldots, |A_k|)$$

⚠️ Bad news about solving equations ⚠️

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- We are still unsure if equations in one single variable, like $AX = B$, can be solved efficiently (conjecture: no)

Reducibility of dynamical systems

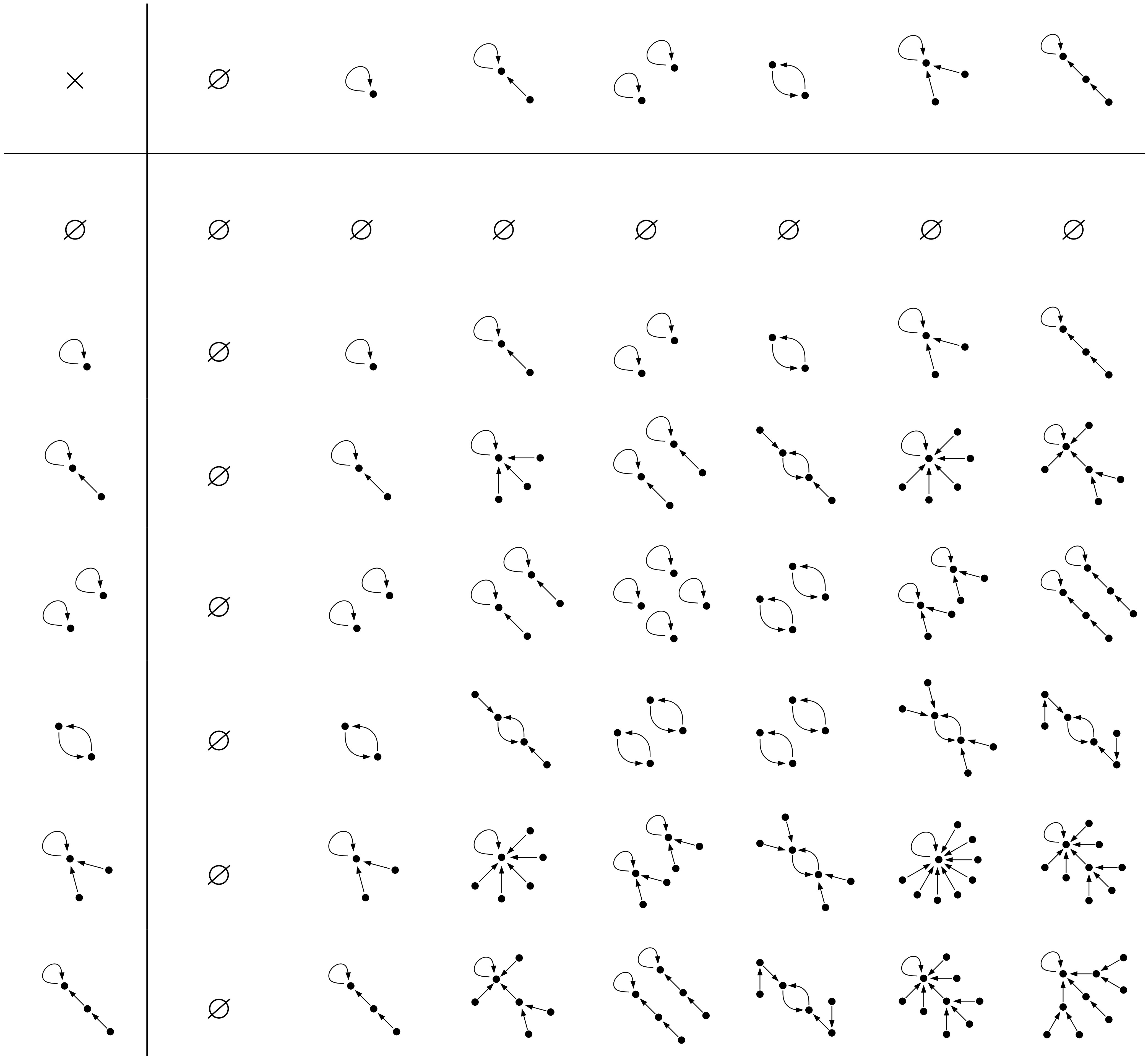
Most dynamical systems are irreducible

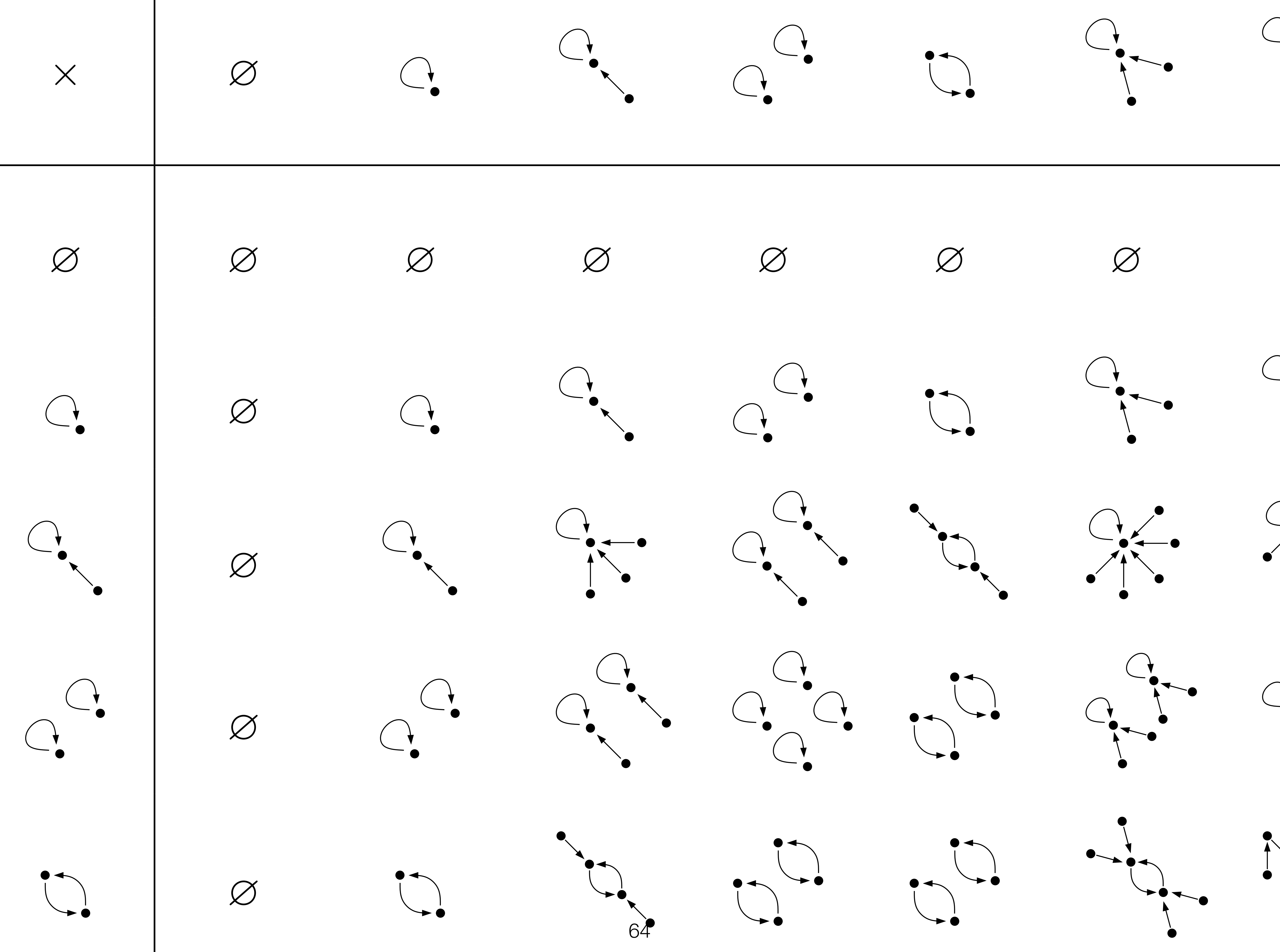
- Formally:

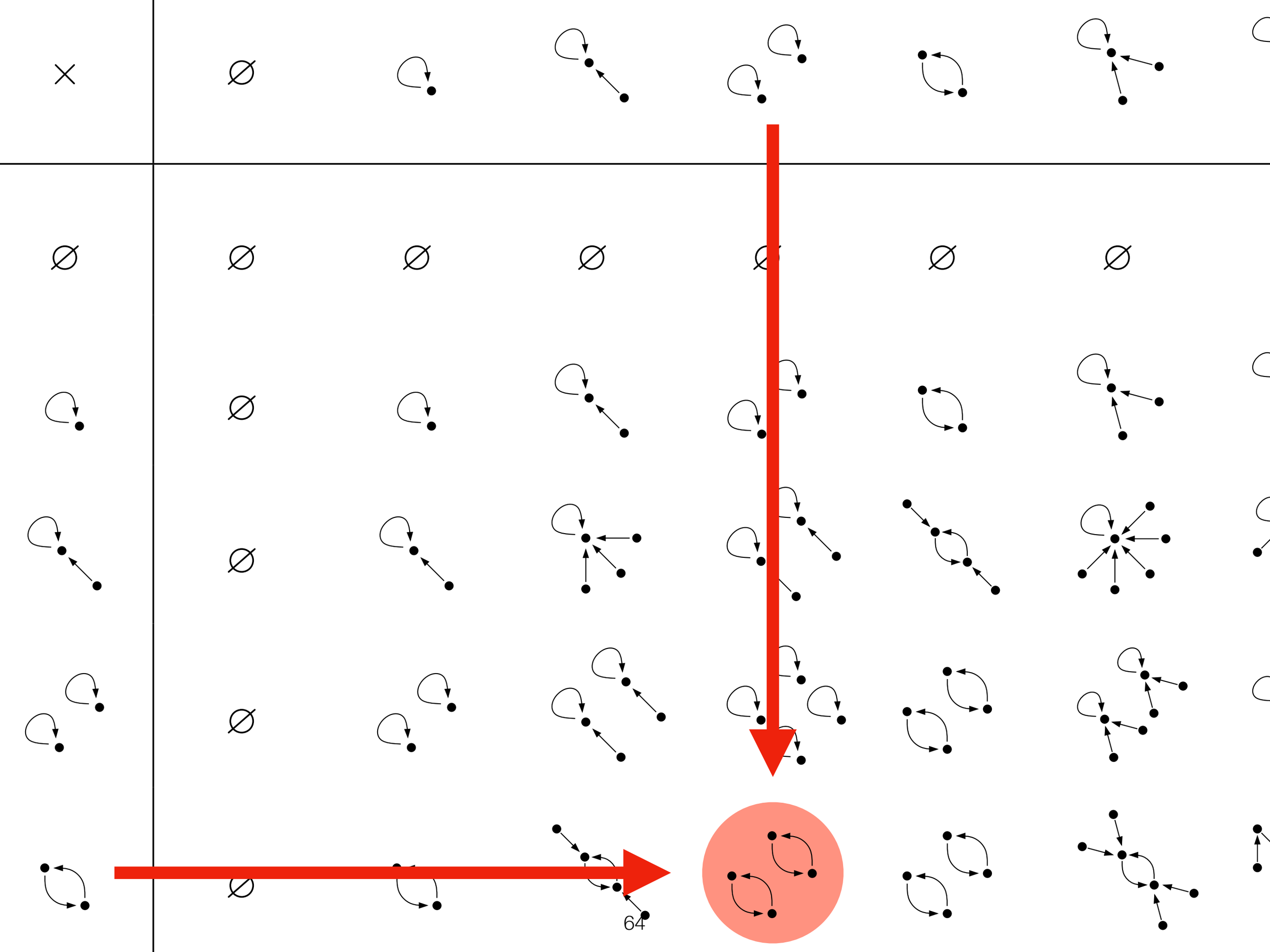
$$\lim_{n\to\infty} \frac{\text{number of reducible systems over} \leq n \text{ states}}{\text{total number of systems over} \leq n \text{ states}} = 0$$

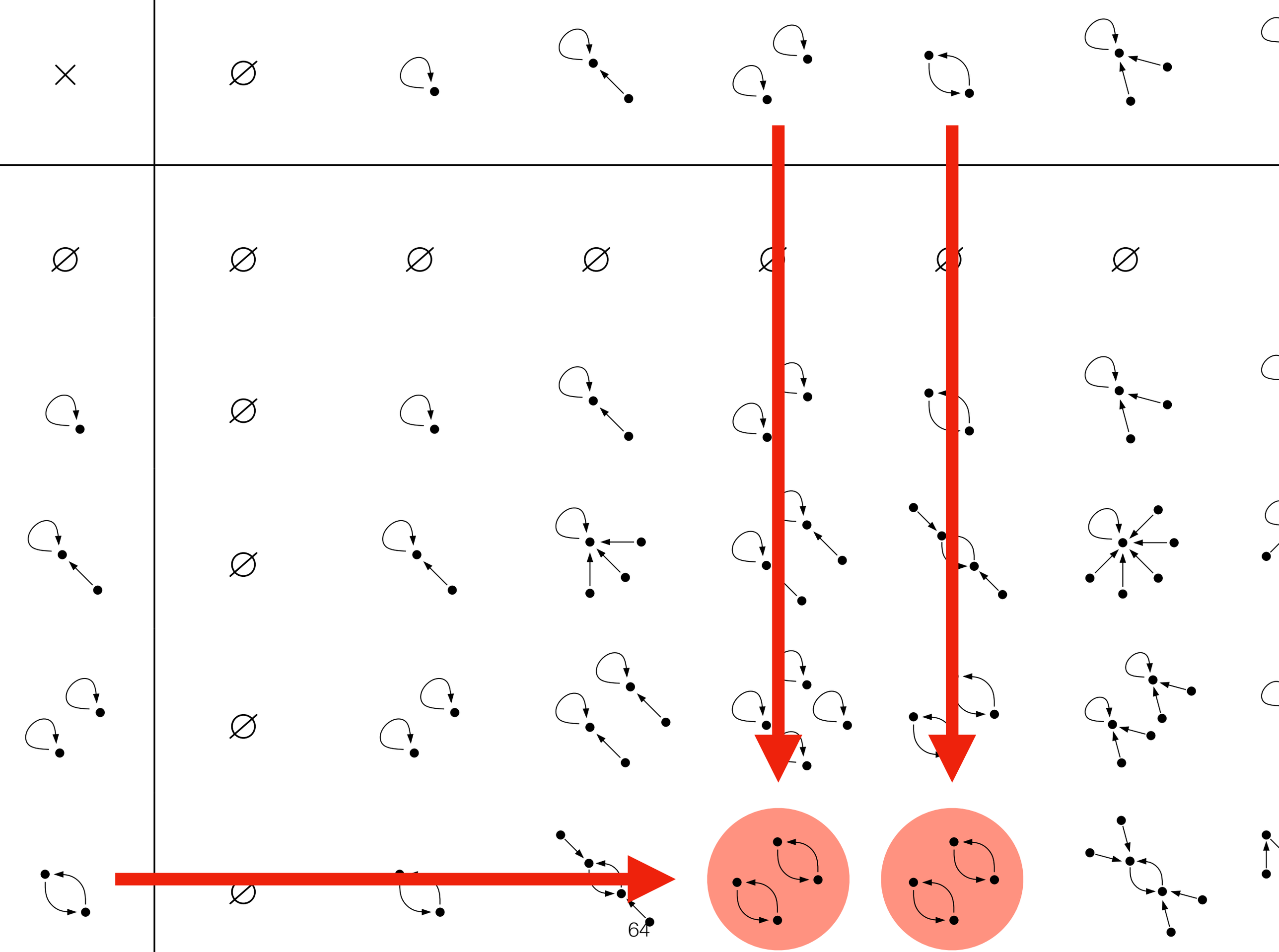
- Notice that this is the opposite of $\mathbb{N}$, where irreducible (aka prime) integers are scarce

No unique factorisation into irreducibles!

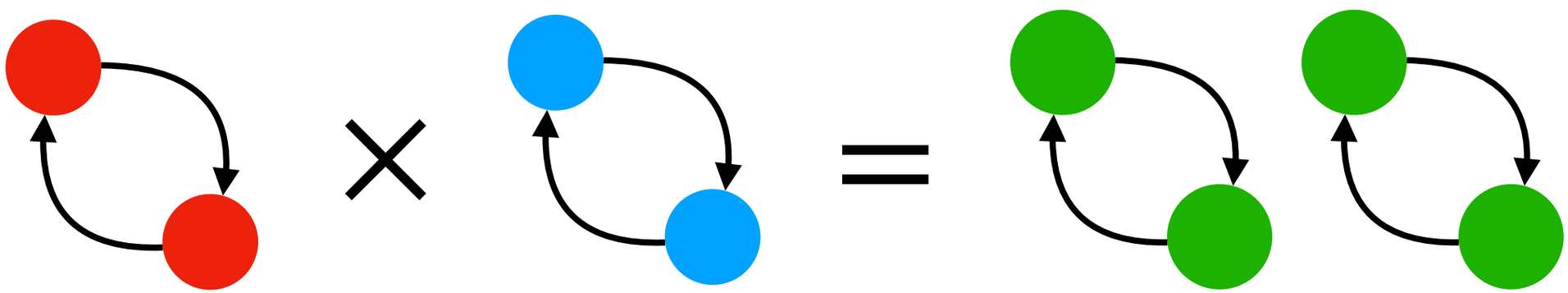


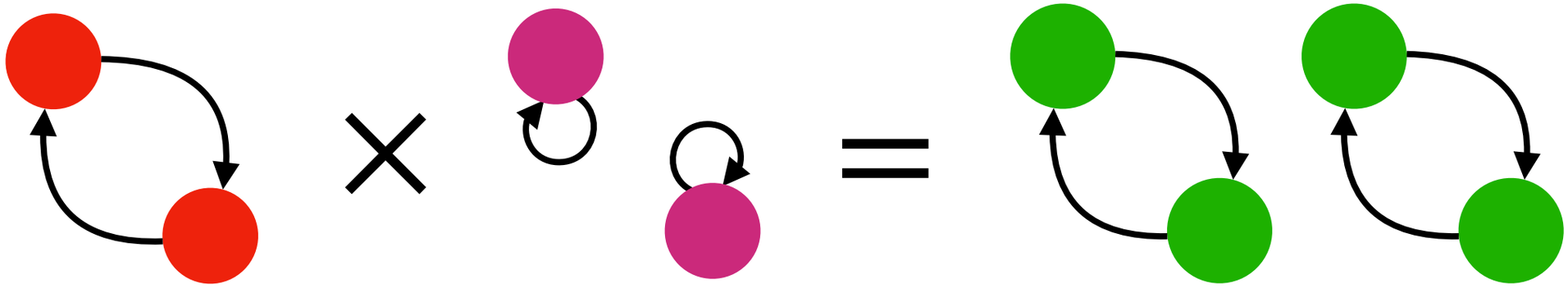






Multiple factorisations





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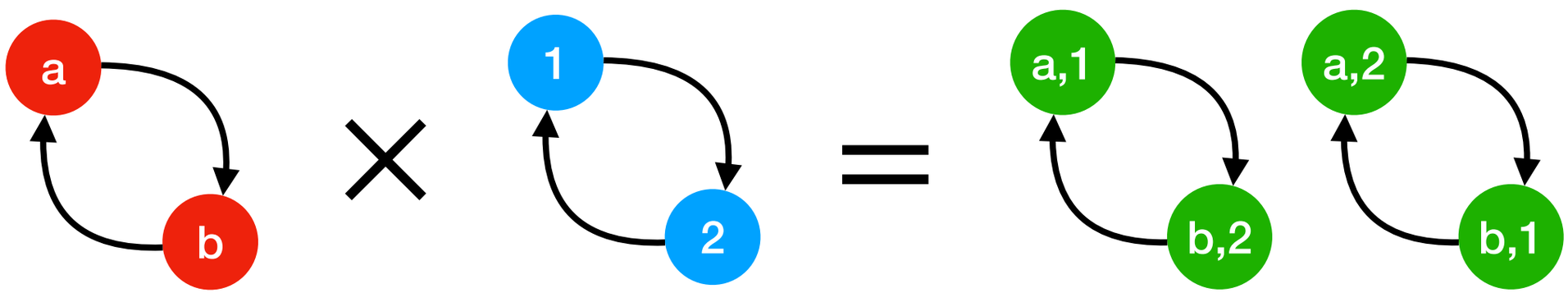

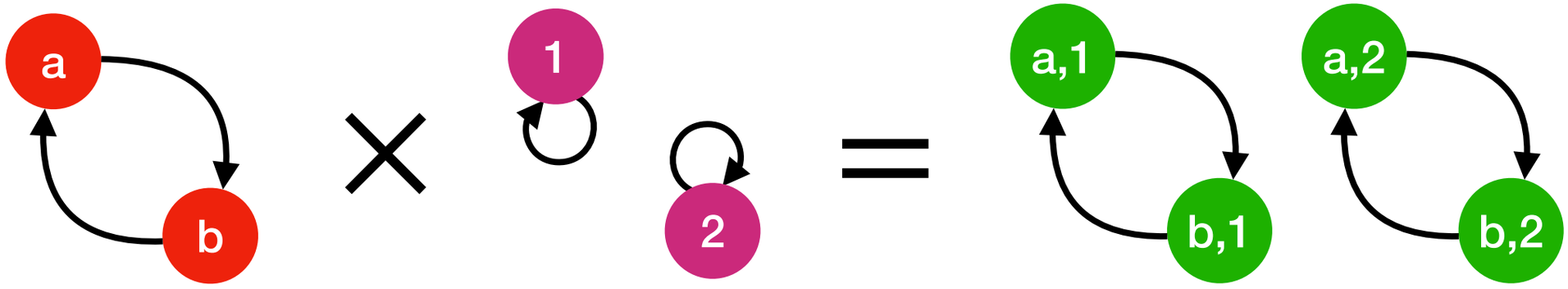

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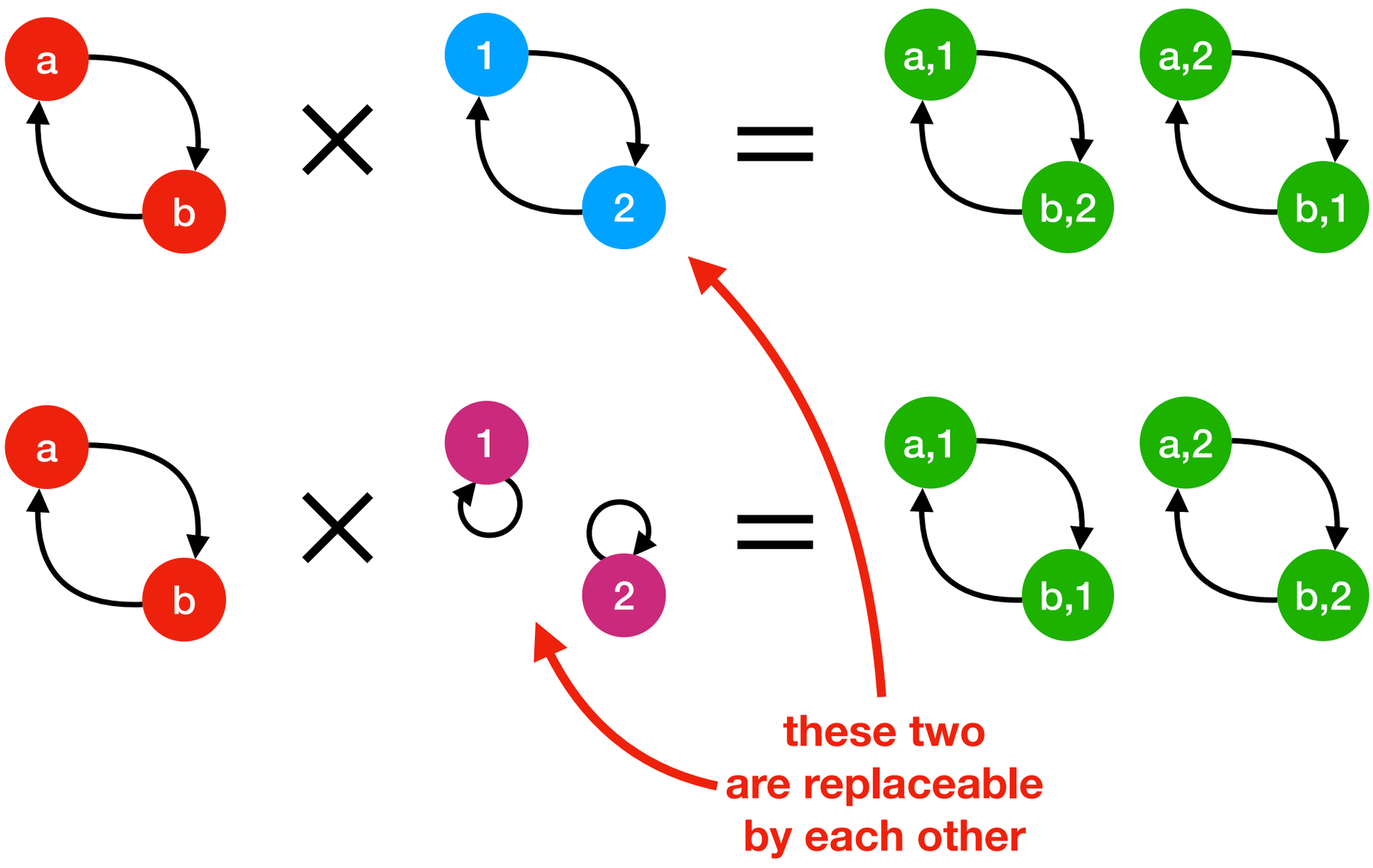

Prime systems

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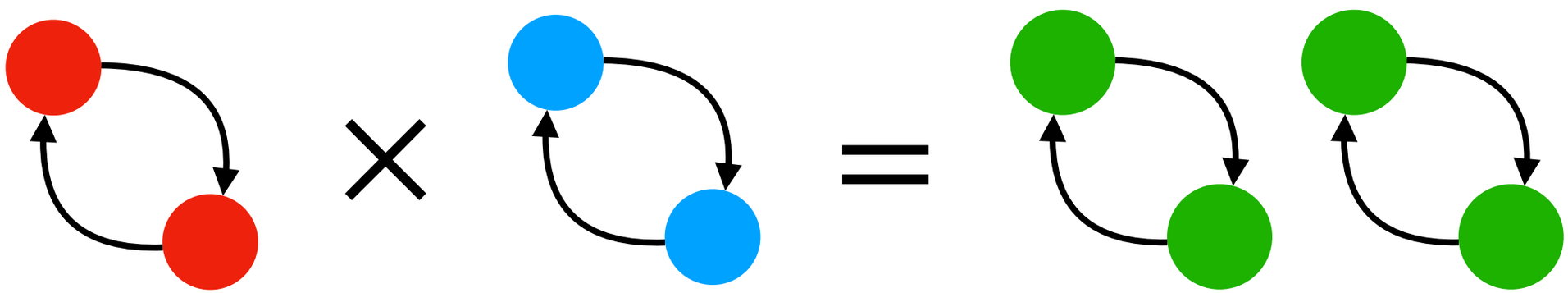
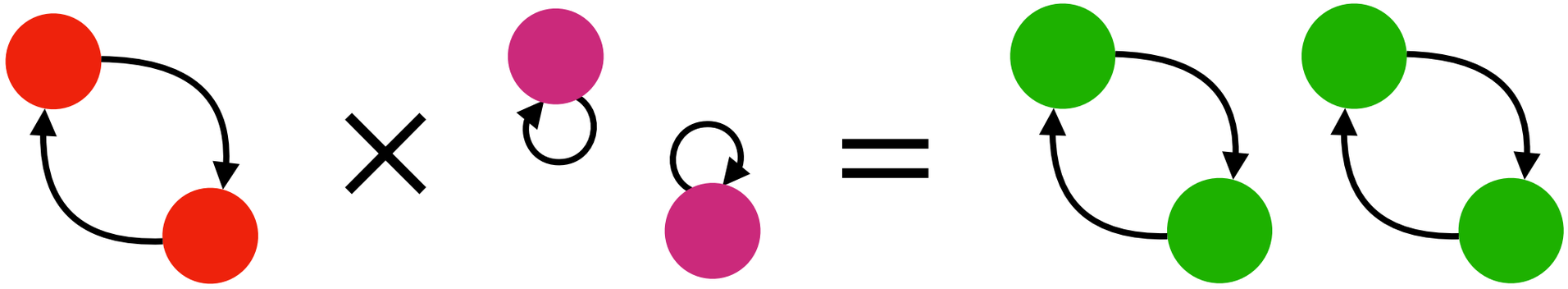
- If a prime appears in one factorisation of a system, then it appears in all the others as well (it is irreplaceable)

An example of nonprime





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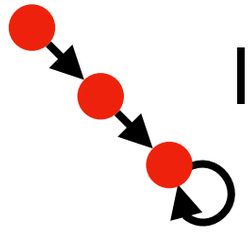
- This guy here?  It took us two years to find out that it is not prime

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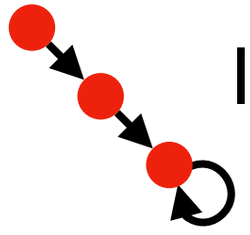
- This guy here?  It took us two years to find out that it is not prime
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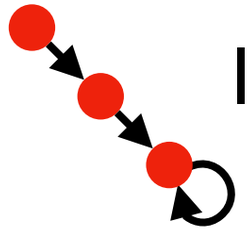
- This guy here?  It took us two years to find out that it is not prime
- A counterexample to the primality of P is two systems A, B such that P divides $A \times B$ but neither A nor B
- Those A and B can be bigger than P, but we don’t know how much, so no algorithm yet…

What do primes look like, if they exist at all?

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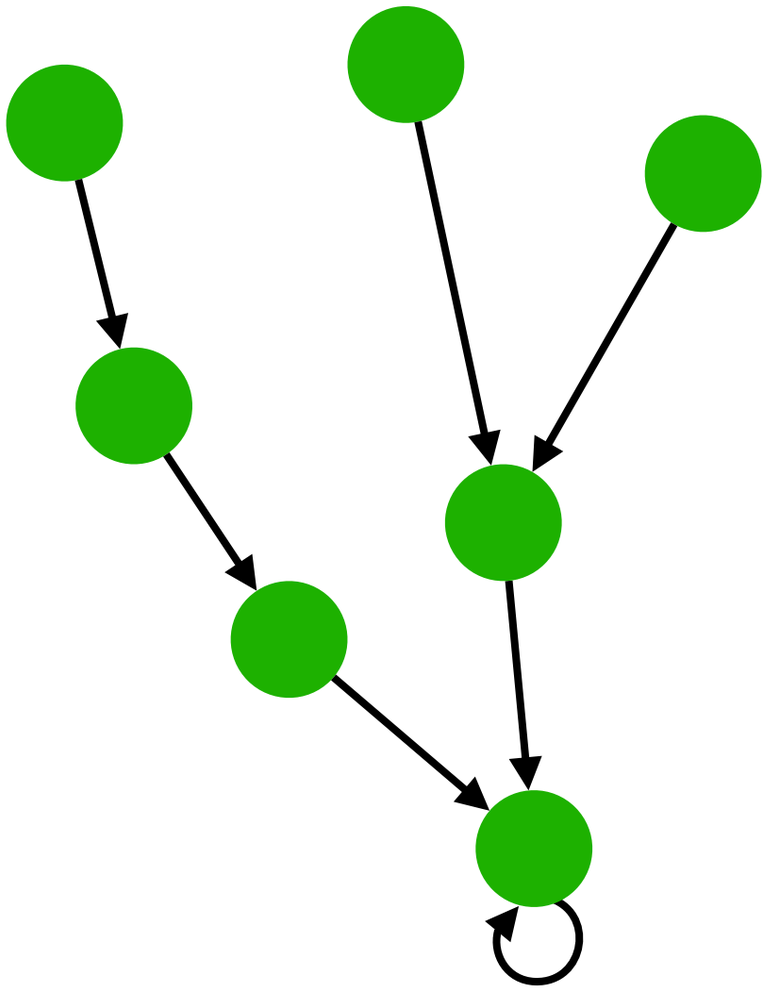

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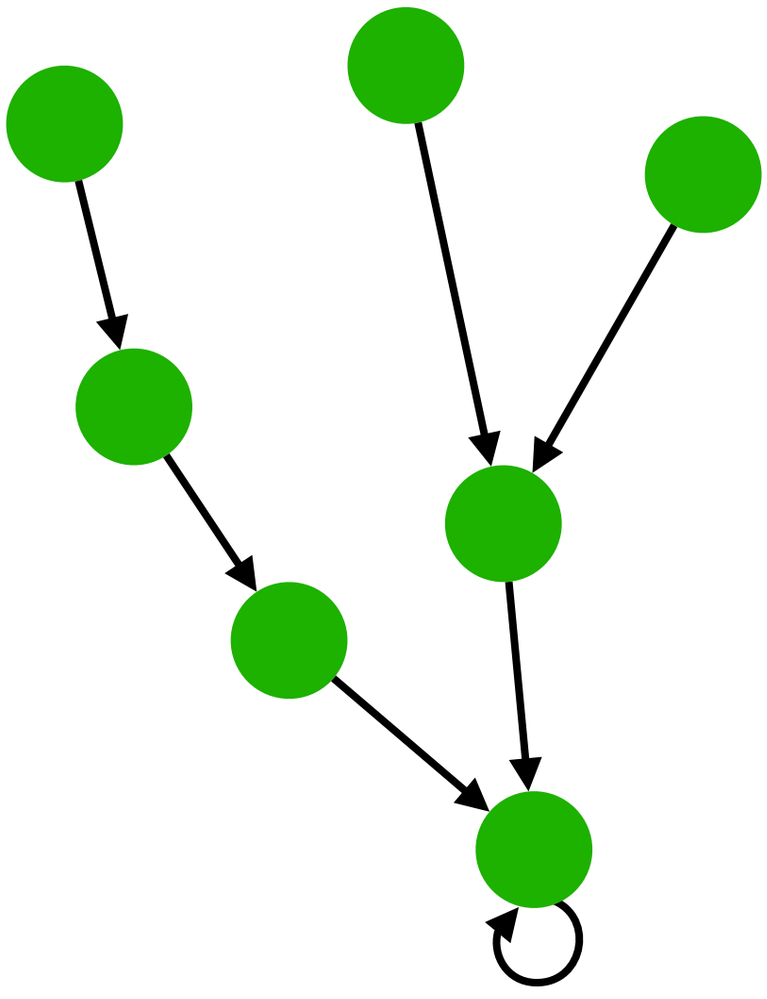
- Connected



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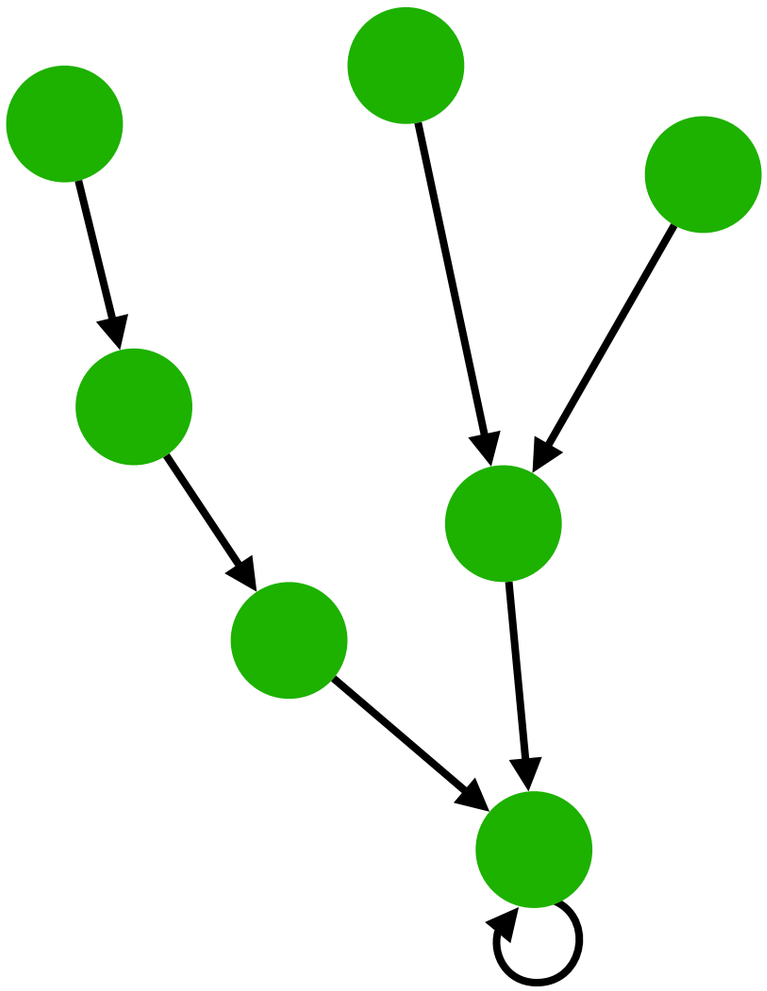
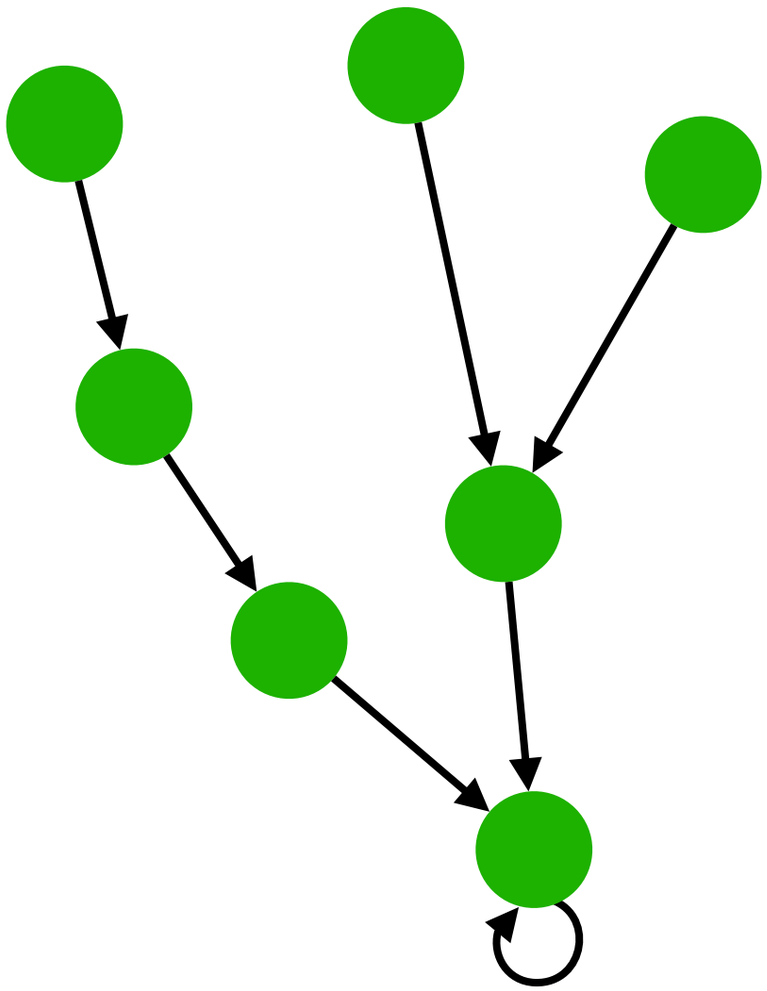
- Fixed point (no cycles of length > 1)



What do primes look like, if they exist at all?

- Connected
- Fixed point (no cycles of length > 1)
- gcd of the number of predecessors across all states must be 1



Future developments

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- Find out if prime systems exist, or at least find a primality algorithm!

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Thanks for your attention!

Merci de votre attention !

Any questions?