Algebraic analysis of discrete dynamical systems

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Finite, discrete-time dynamical systems

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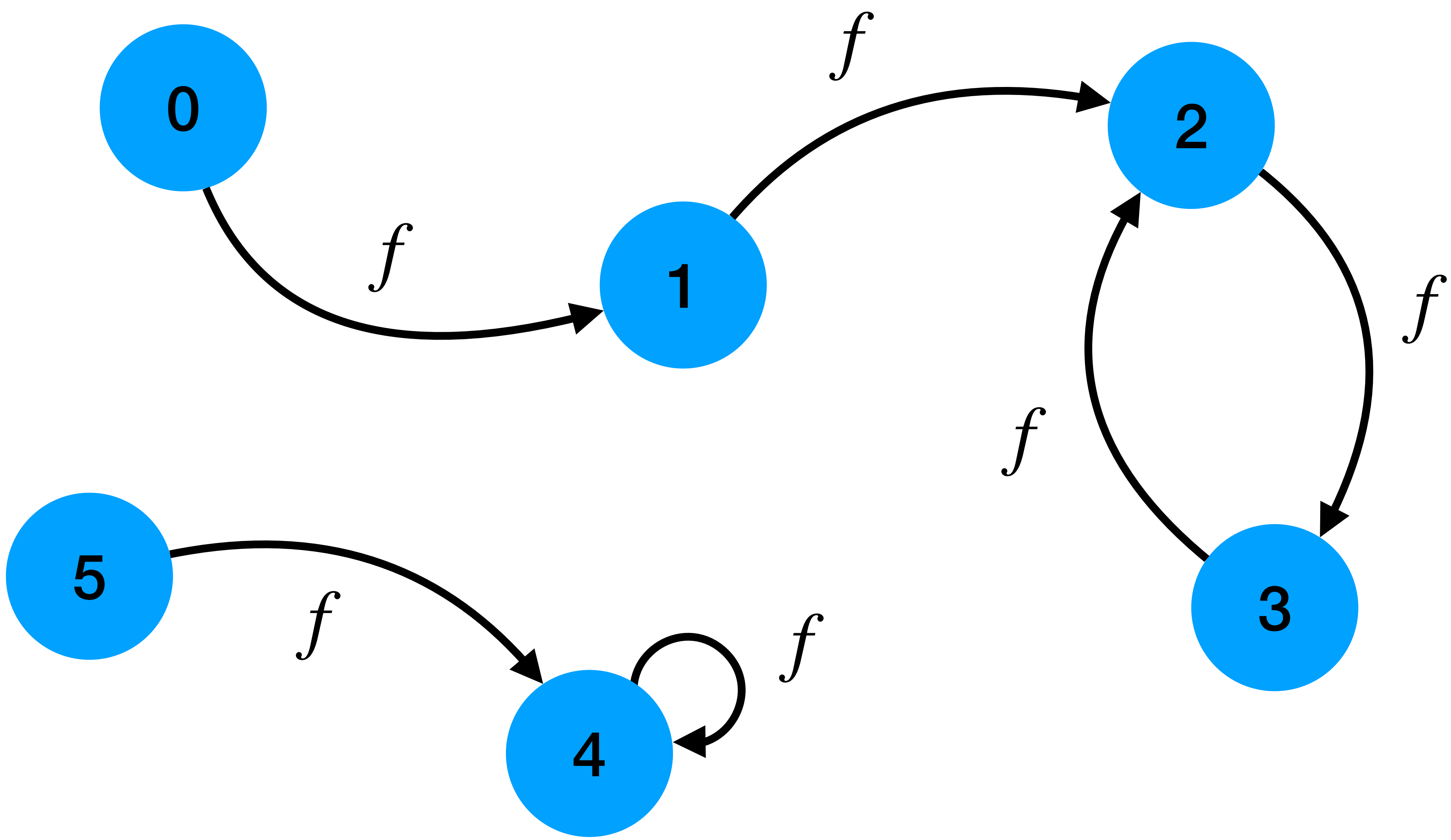
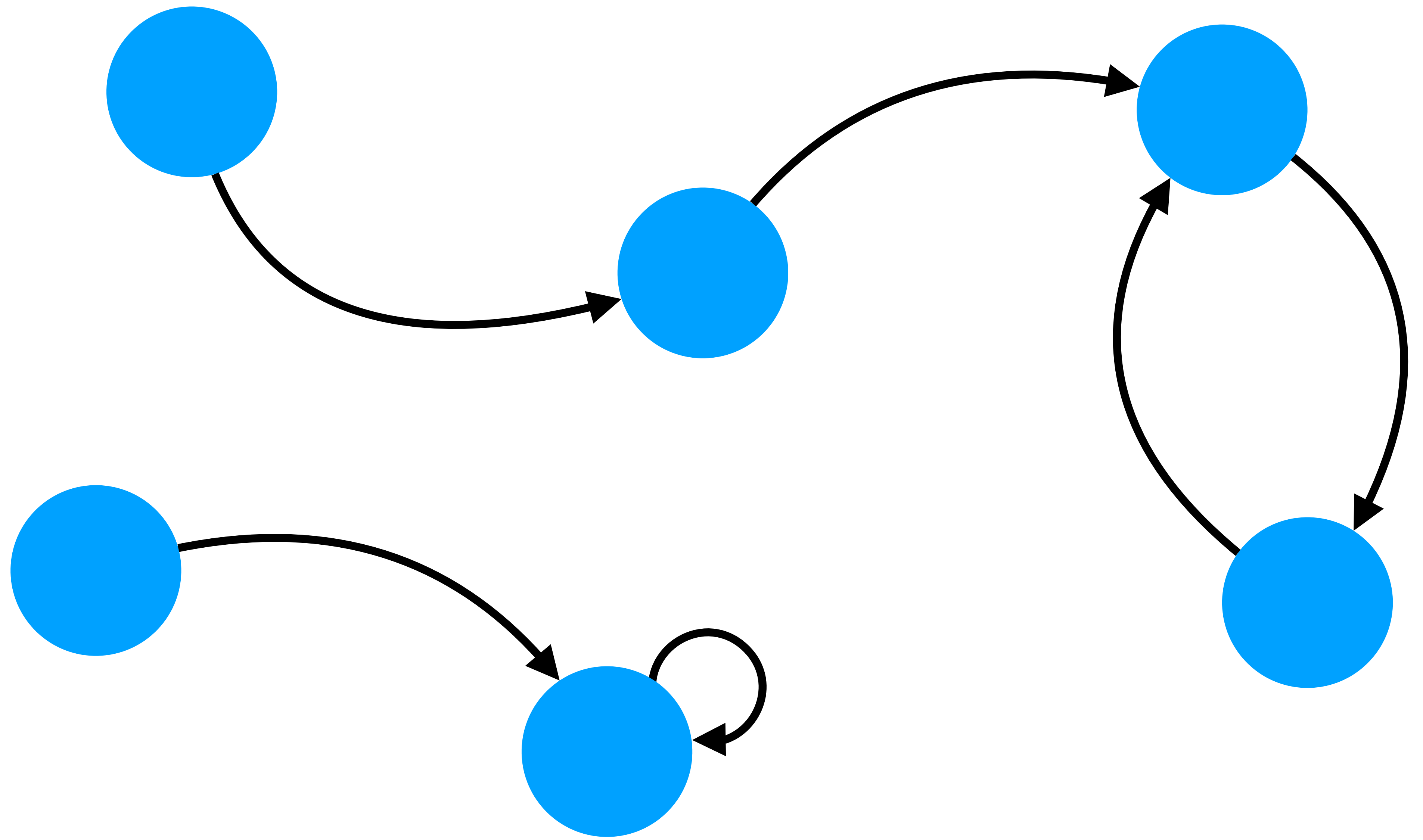
Just a finite set with a transition function (A, f)



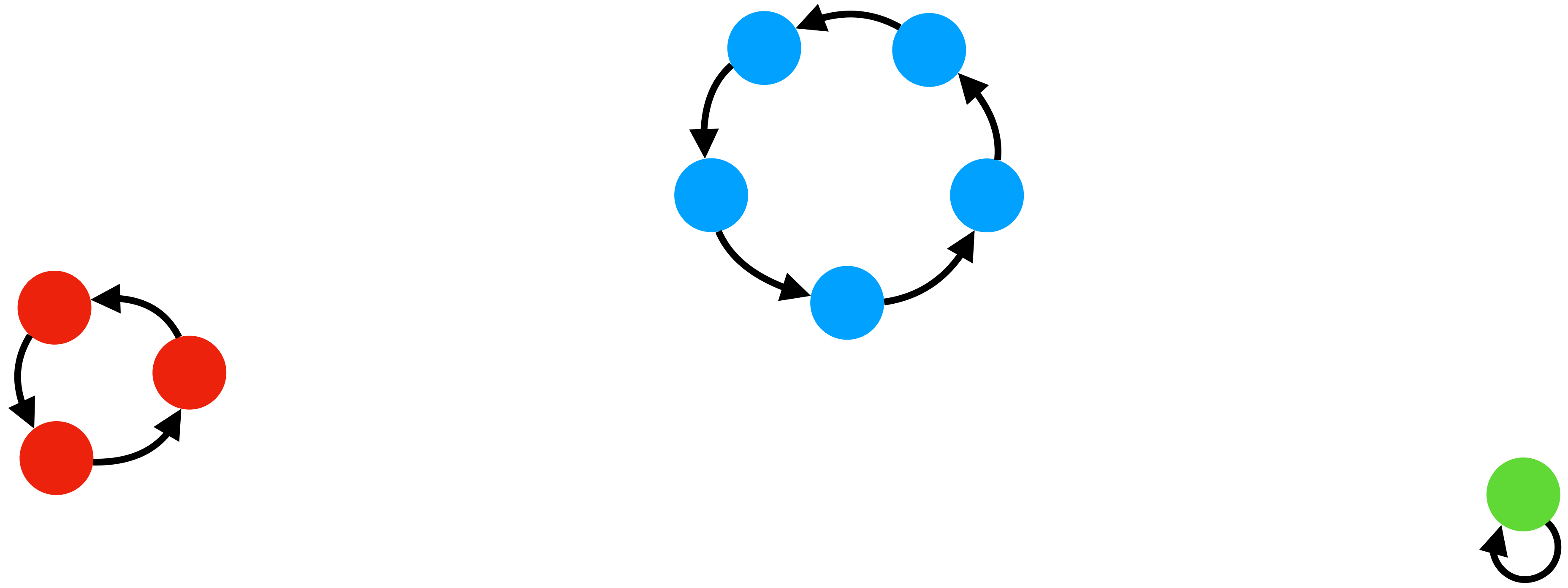
Finite, discrete-time dynamical systems

Just a finite set with a transition function (A, f) modulo isomorphism



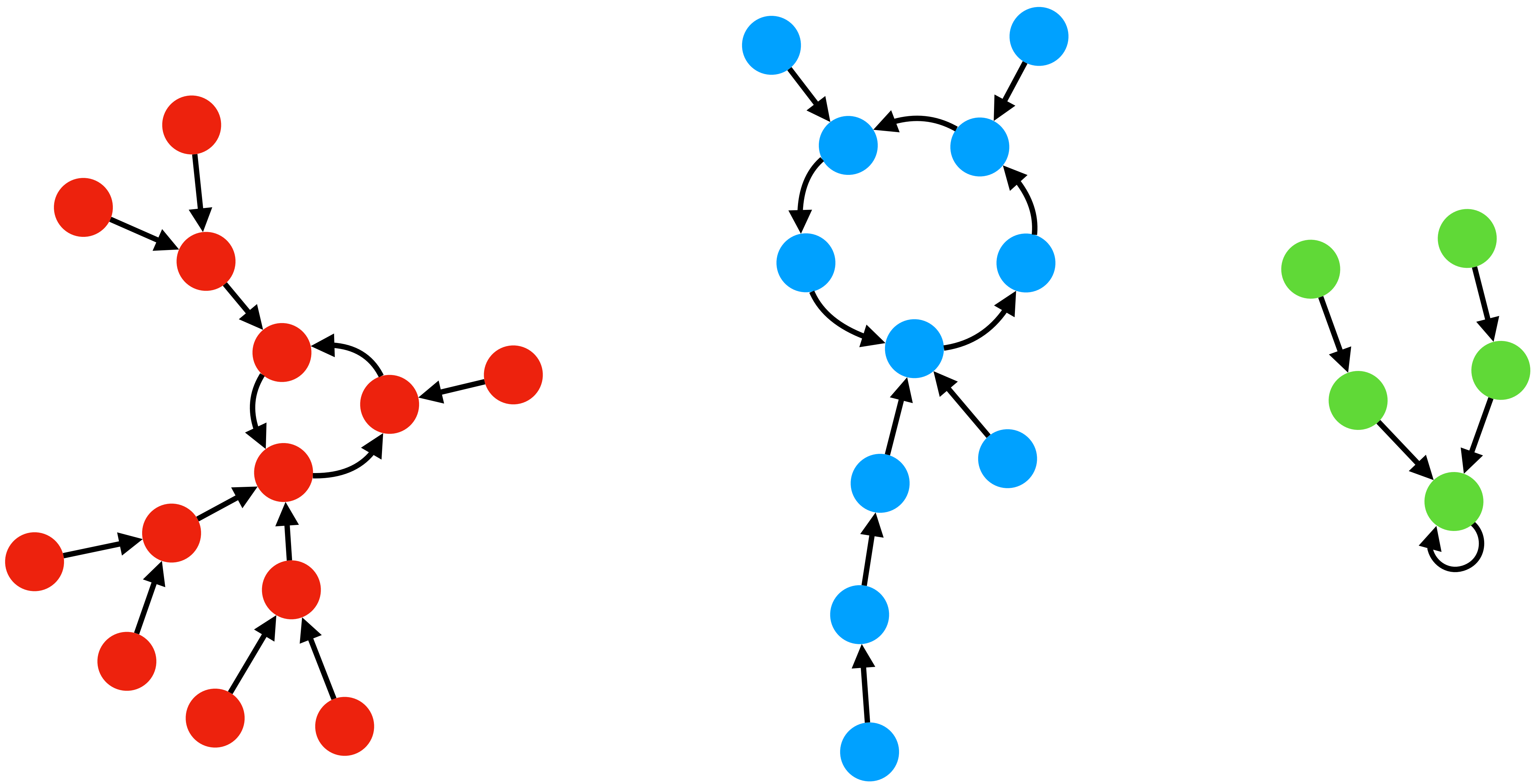
General shape of a dynamical system

A few limit cycles



General shape of a dynamical system

A few limit cycles with trees going in



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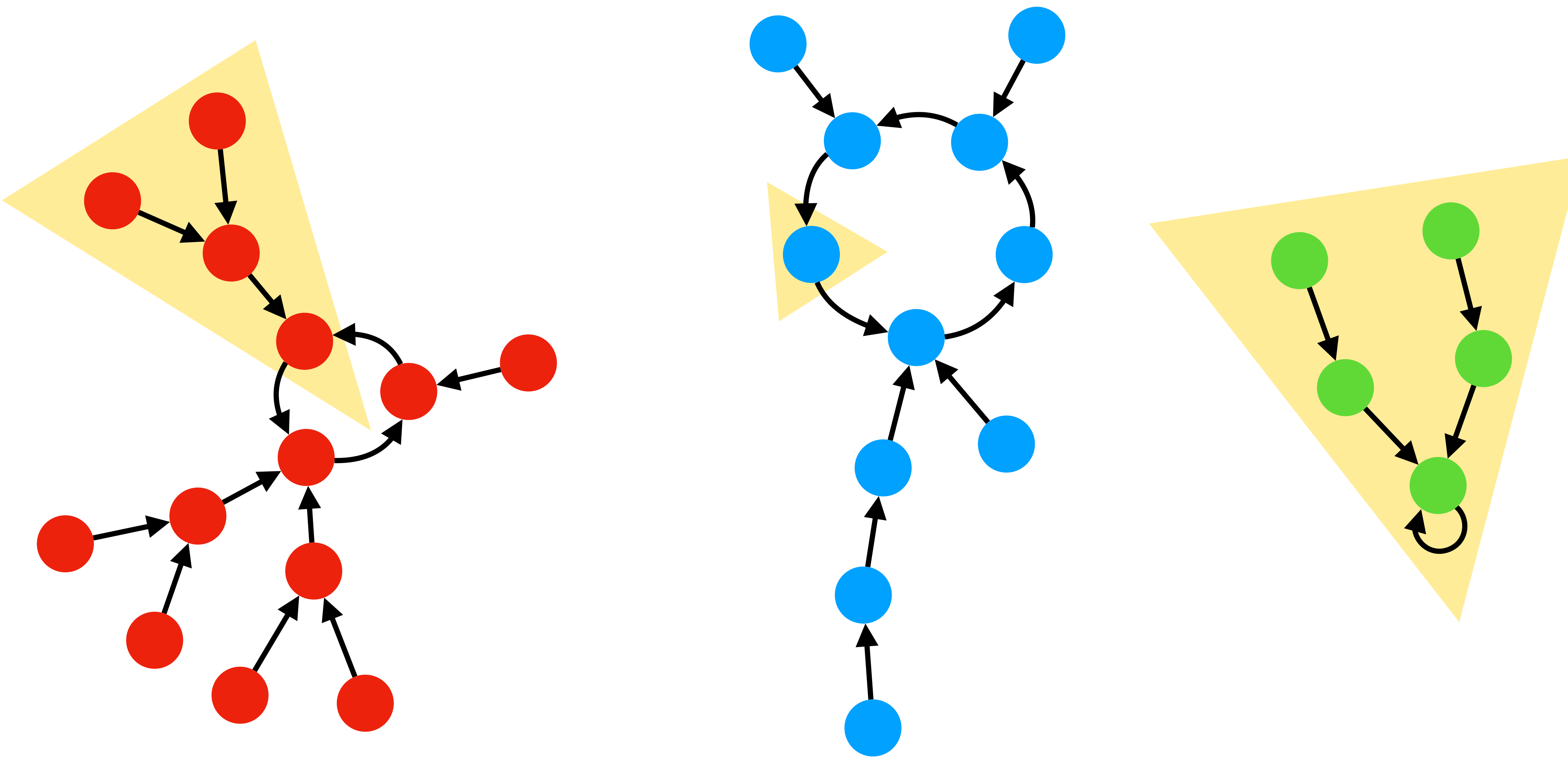

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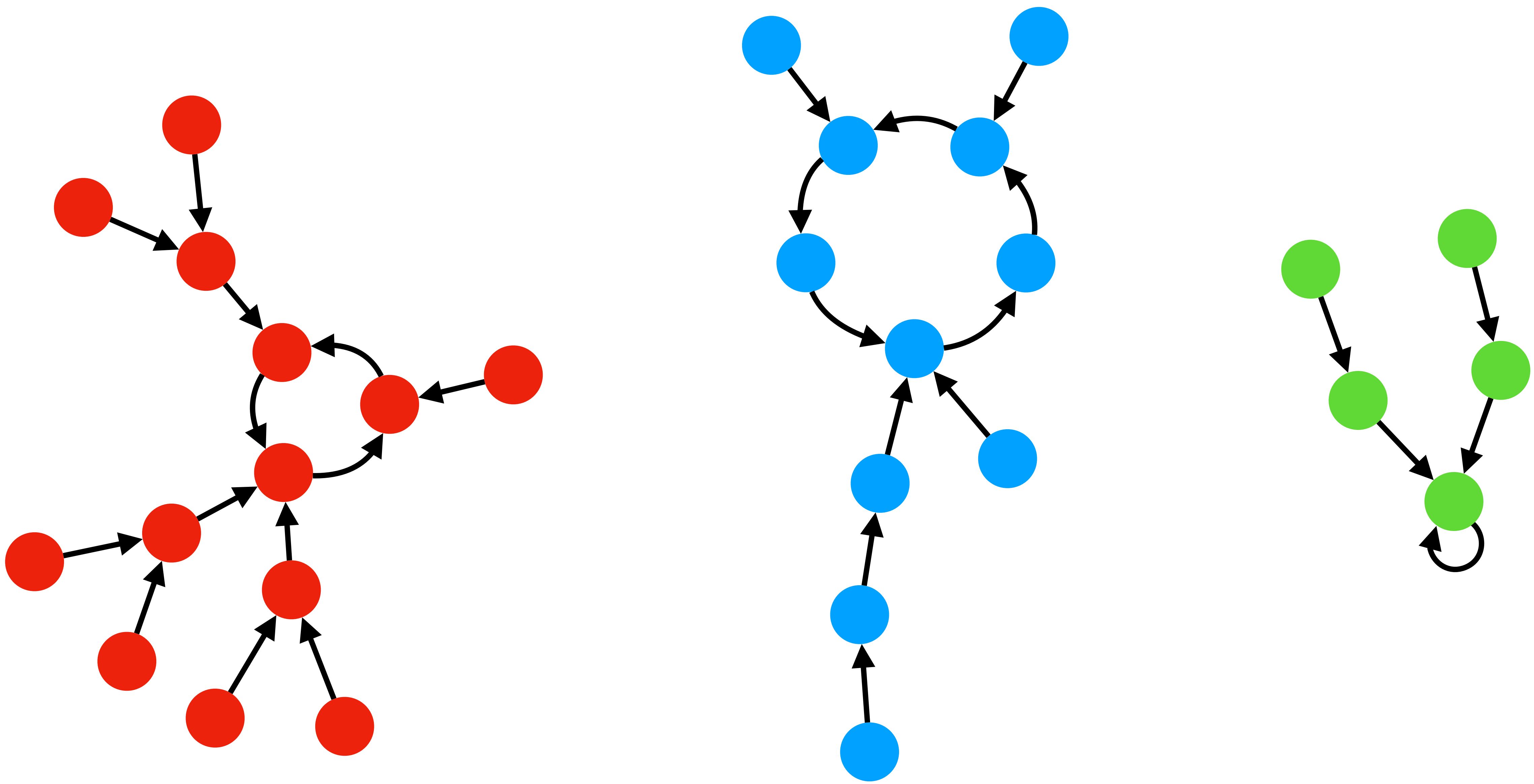

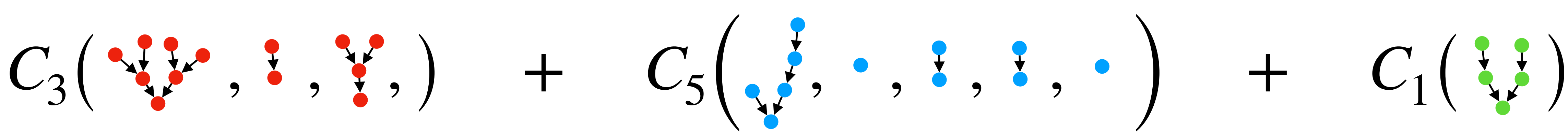

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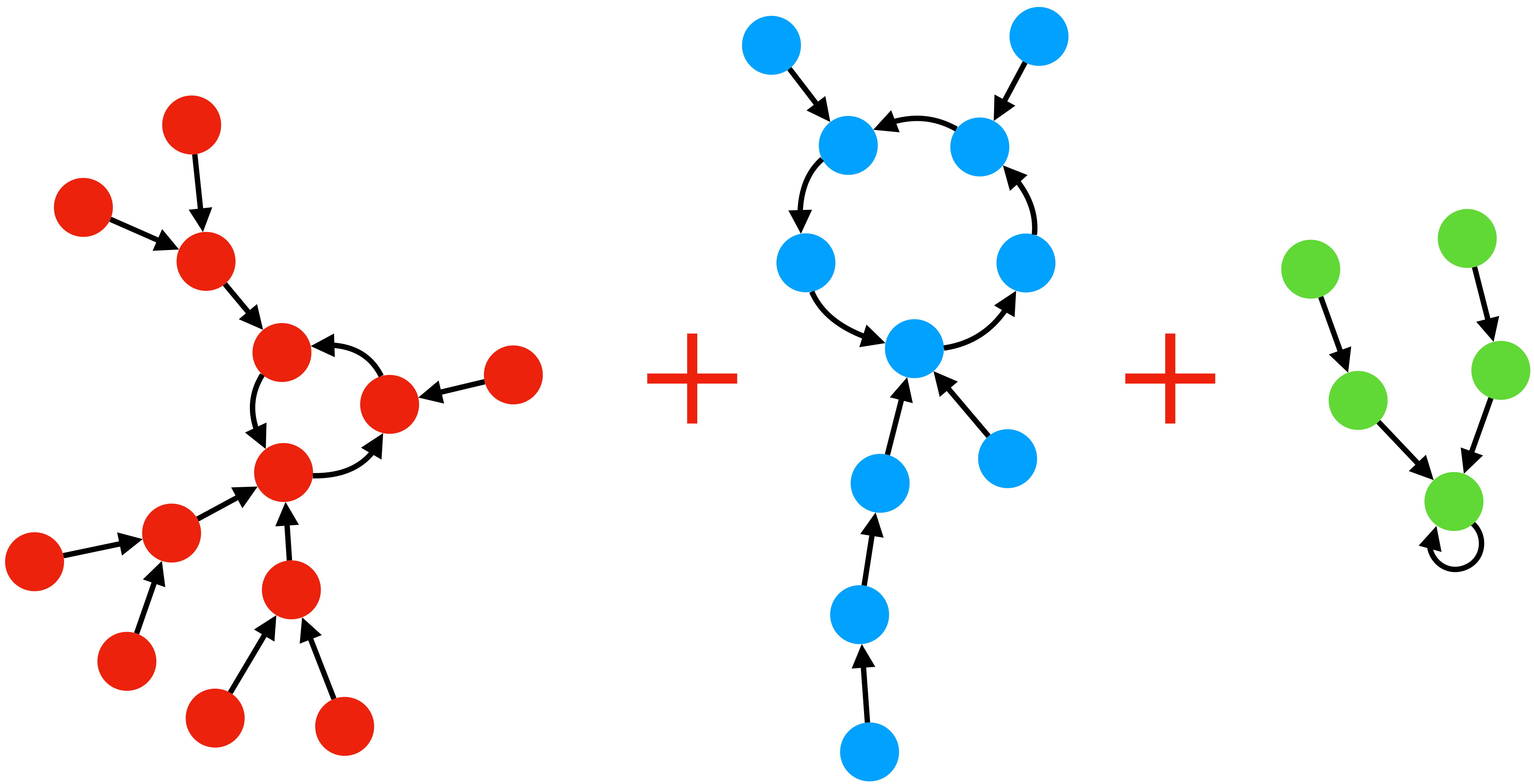
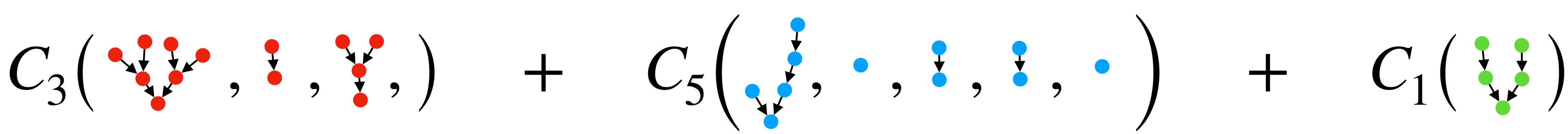
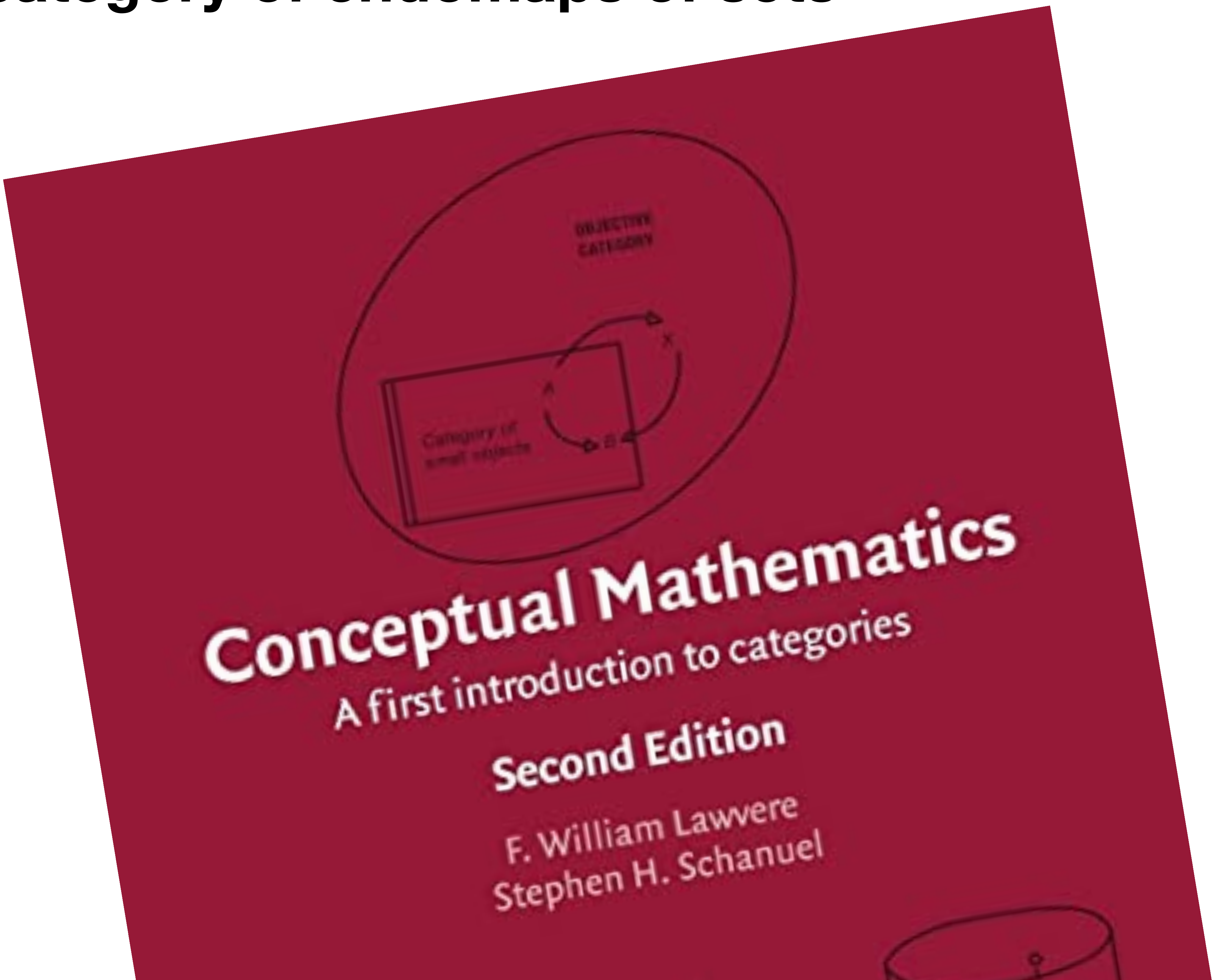




Operations over dynamical systems

The inspiration

The category of endomaps of sets



Sum of dynamical systems

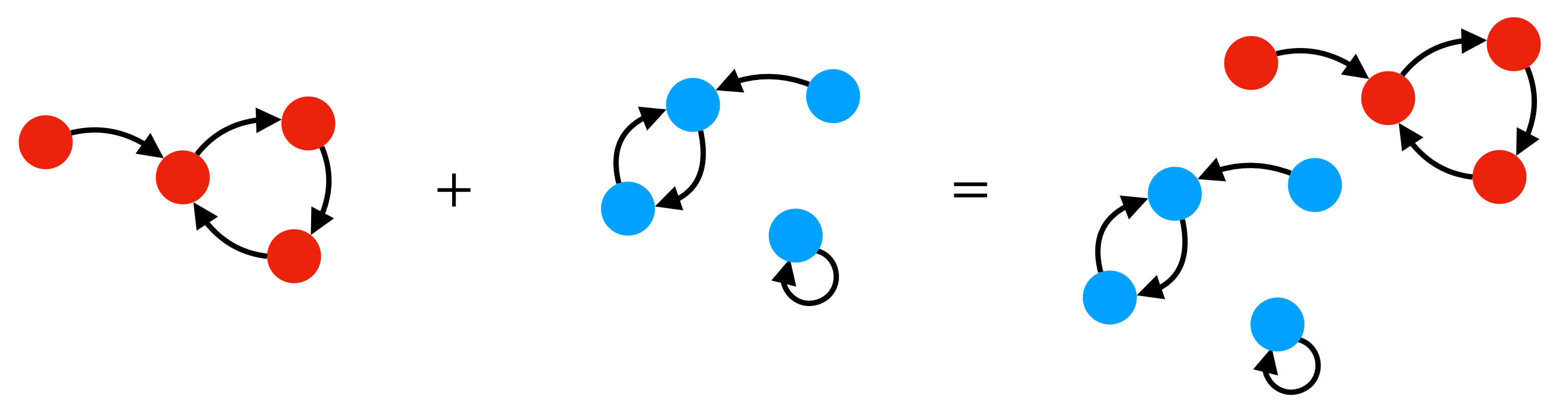
Necessary but not that interesting

- In graph-theoretic terms, it's just the disjoint union

$$(A, f) + (B, g) = (A \uplus B, f + g) \quad \text{with } (f+g)(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

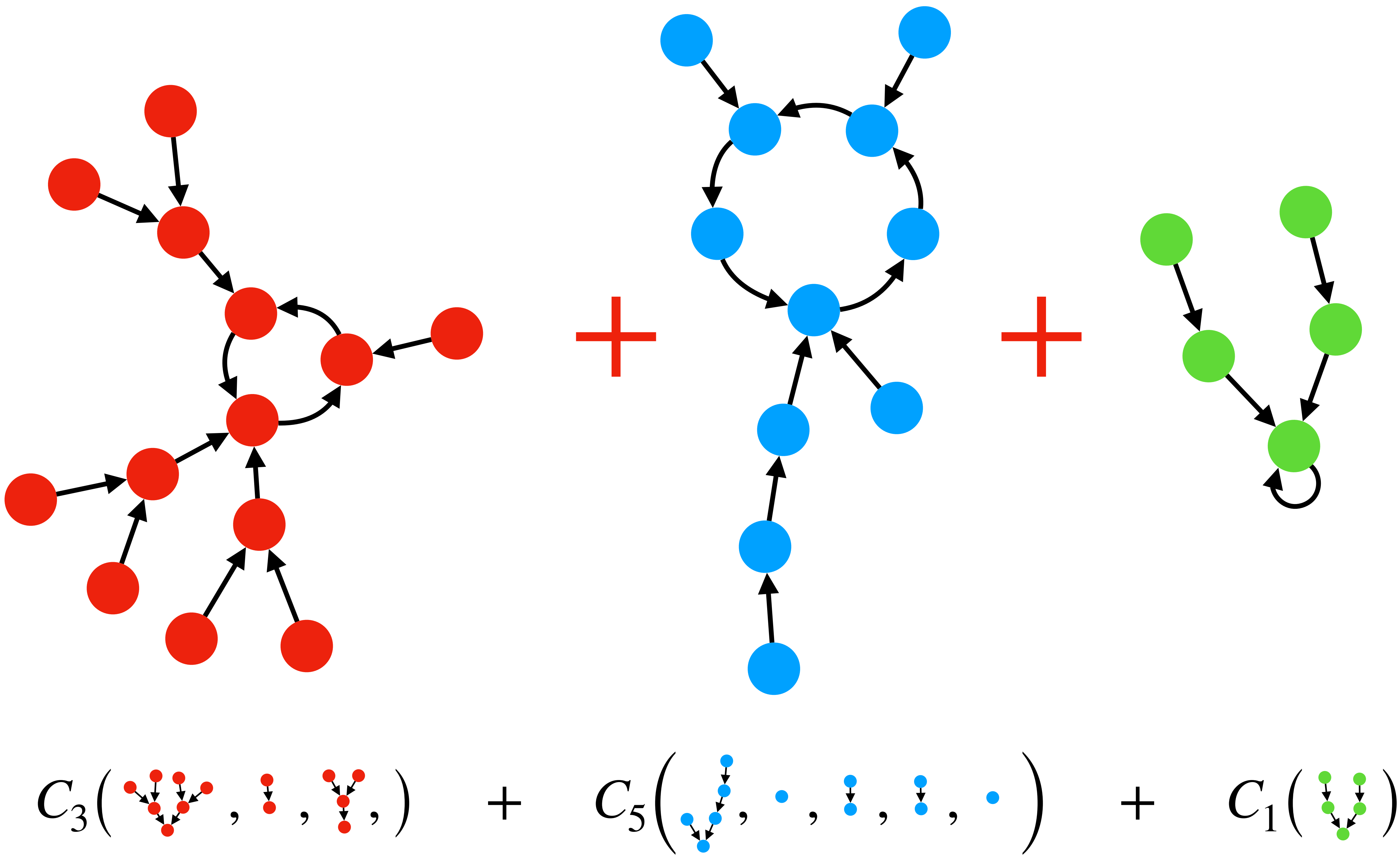
- This represents the alternative execution of A and B

- The identity is the empty system $\mathbf{0} = (\varnothing, \varnothing)$



General shape of a dynamical system

It’s a sum of cycles with trees going in



Product of dynamical systems

Now we're talking!

- In graph-theoretic terms, it's the tensor product

$$(A, f) \times (B, g) = (A \times B, f \times g)$$

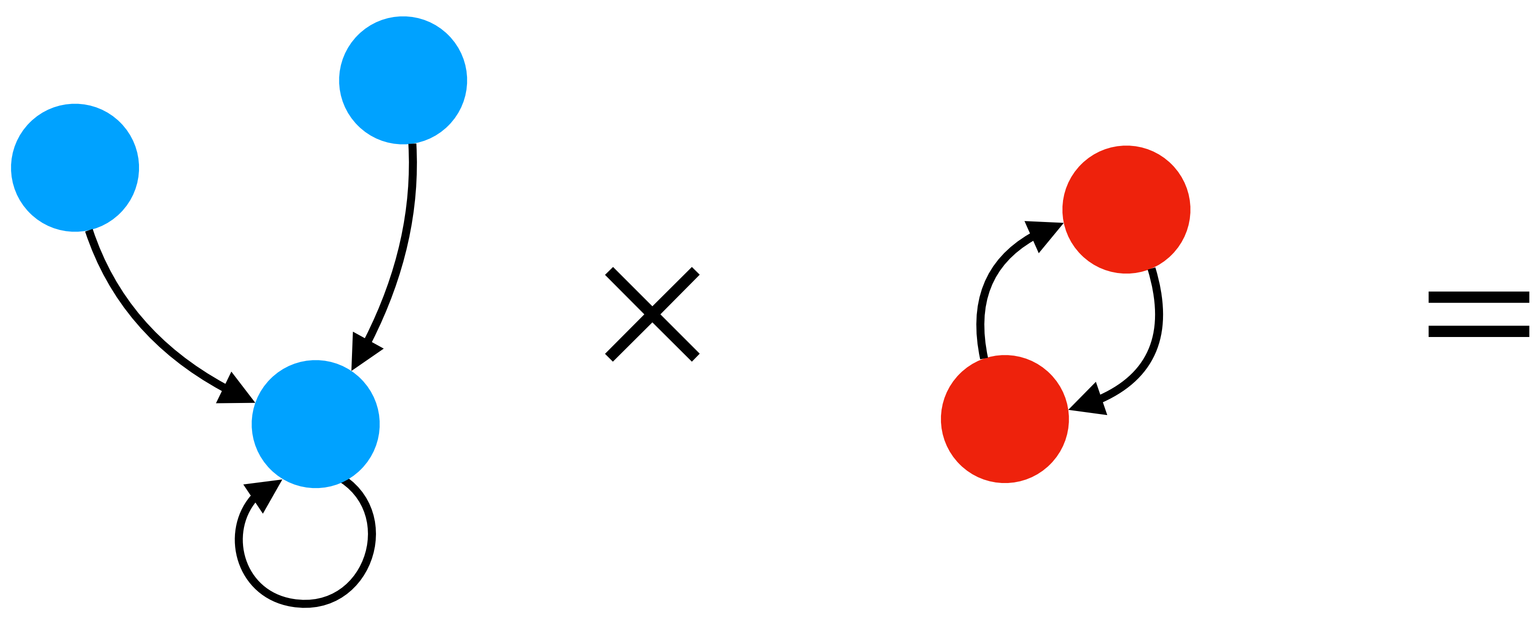
$$\text{with } (f \times g)(a, b) = (f(a), g(b))$$

- This represents the synchronous execution of A and B

- The identity is the singleton system $\mathbf{1} = (\{0\}, \mathrm{id})$

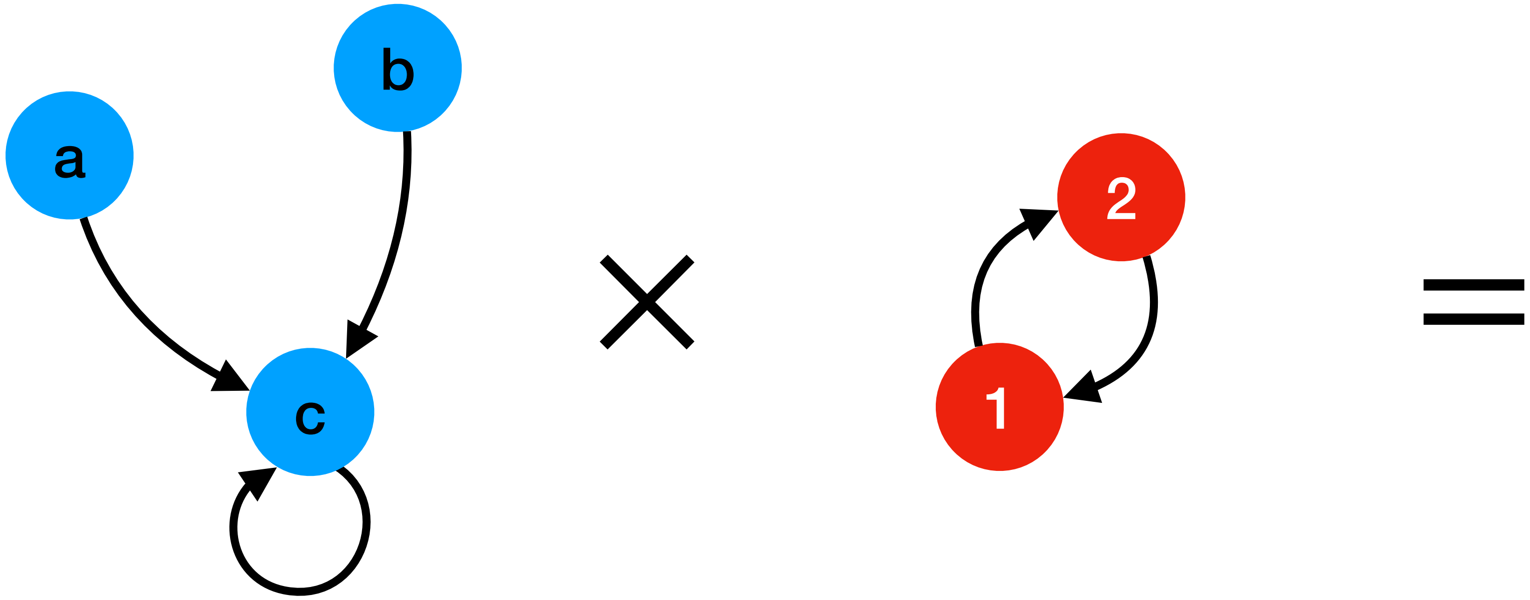
Product in $\mathbf{D}$ is graph tensor product

Two systems modulo isomorphism



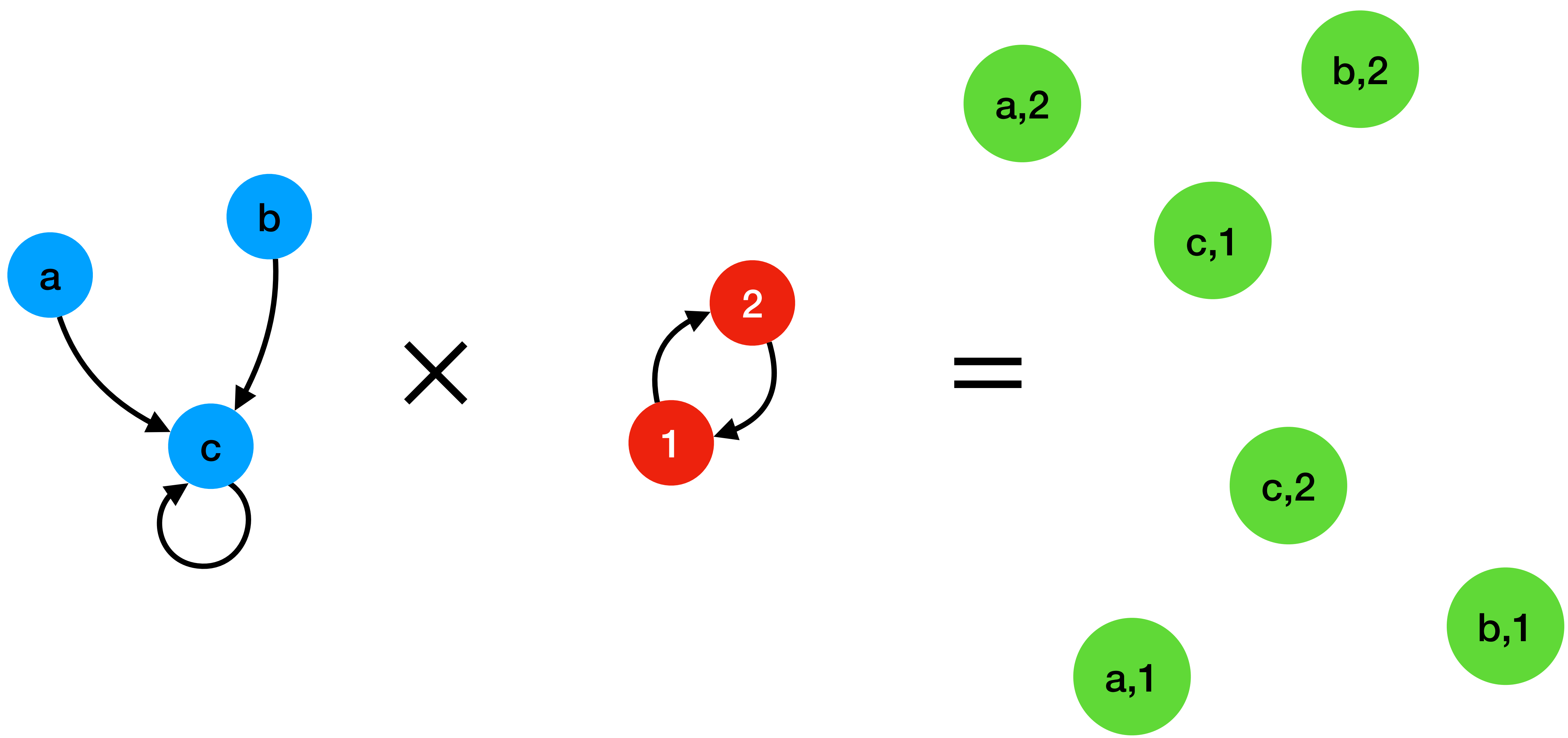
Product in $\mathbf{D}$ is graph tensor product

Temporary state names



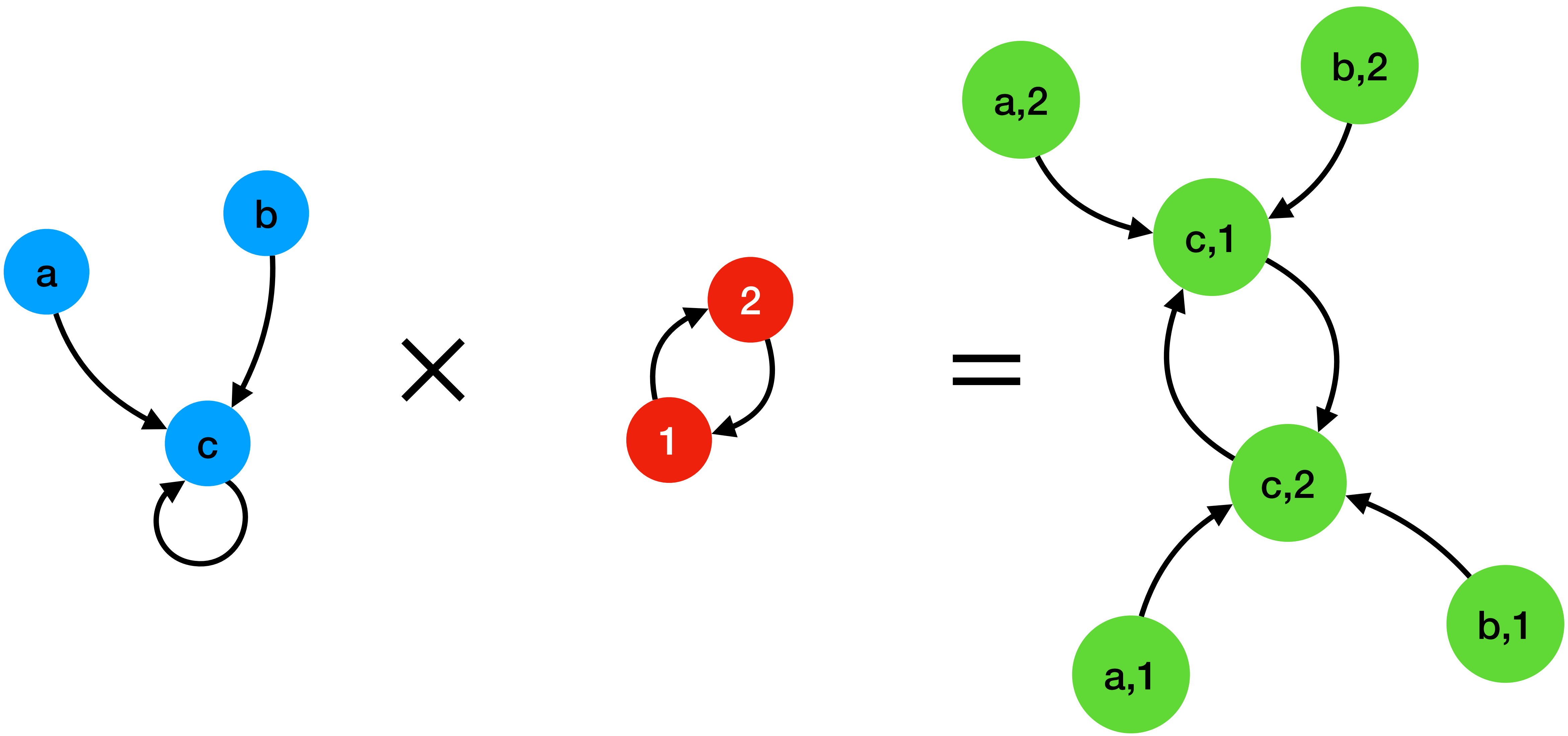
Product in $\mathbb{D}$ is graph tensor product

Cartesian product of the states



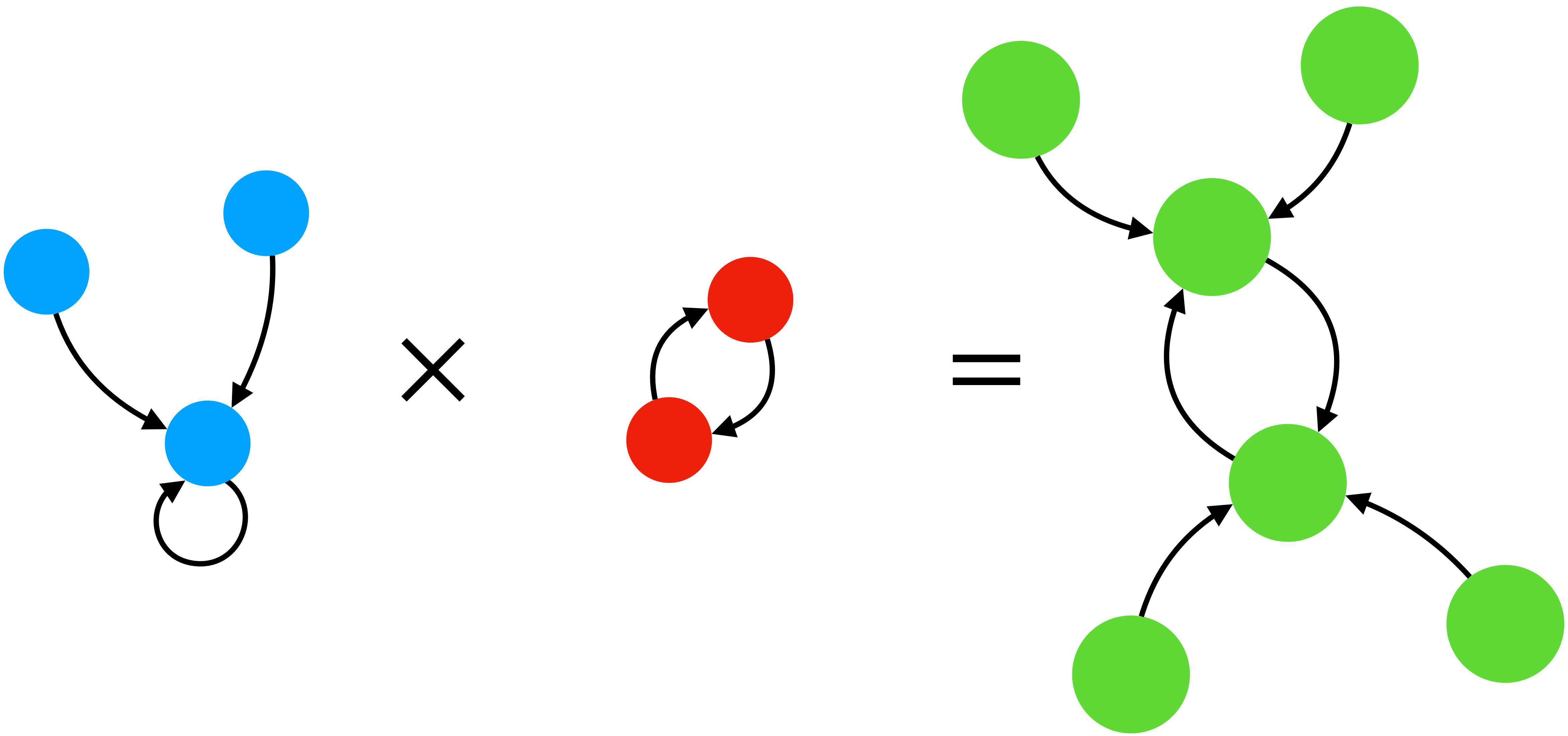
Product in $\mathbf{D}$ is graph tensor product

Arrows iff arrows between both components

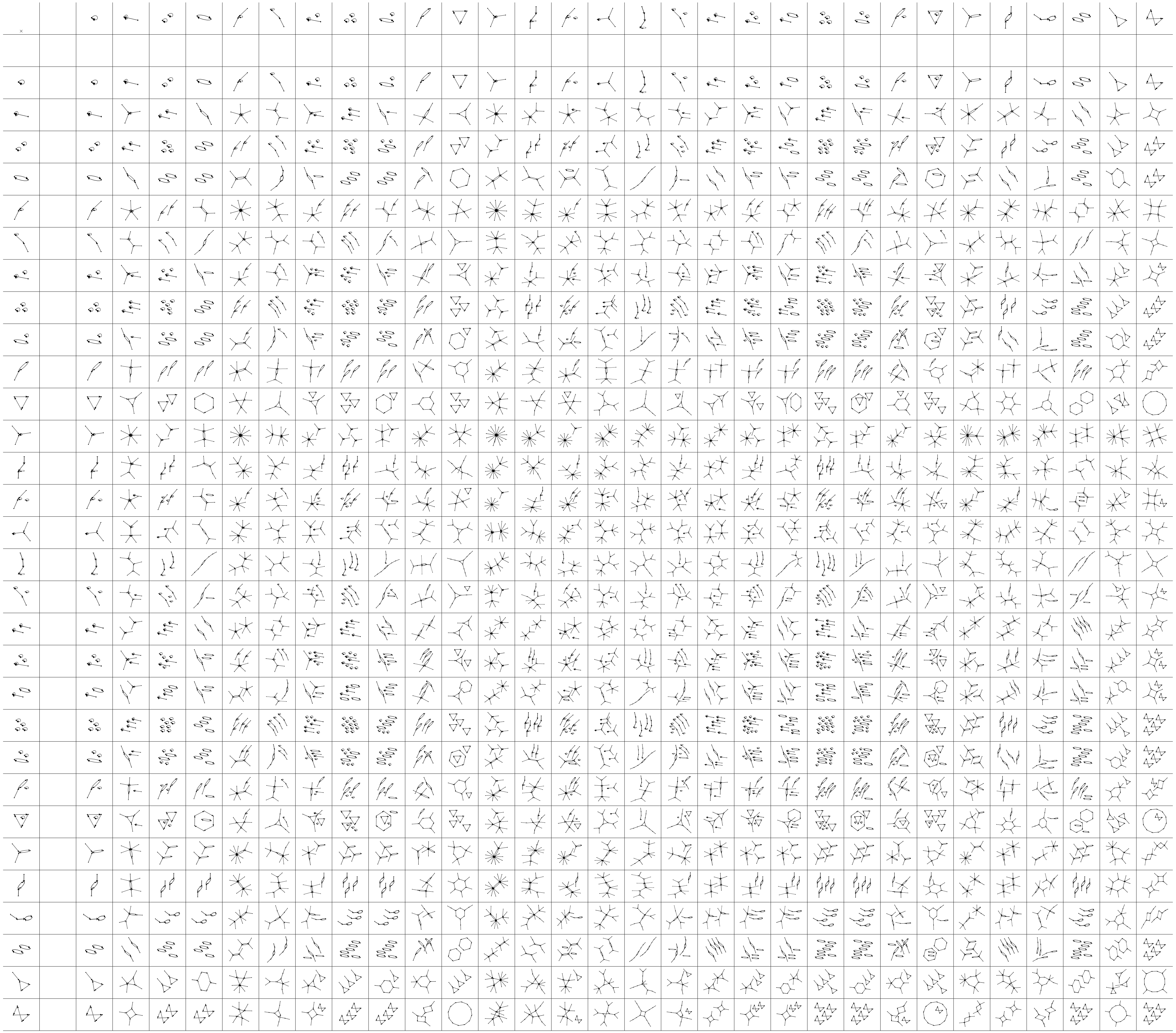


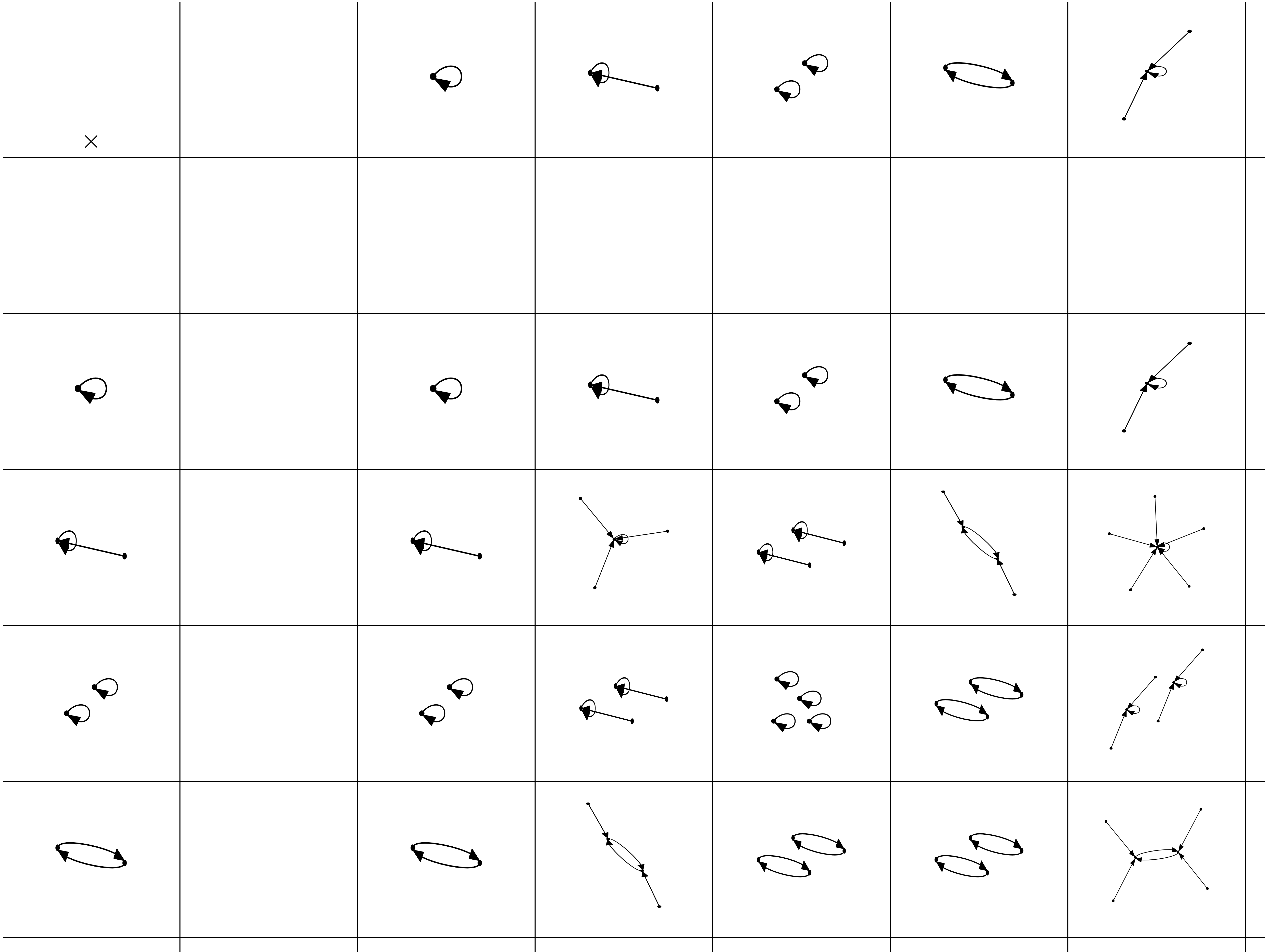
Product in $\mathbb{D}$ is graph tensor product

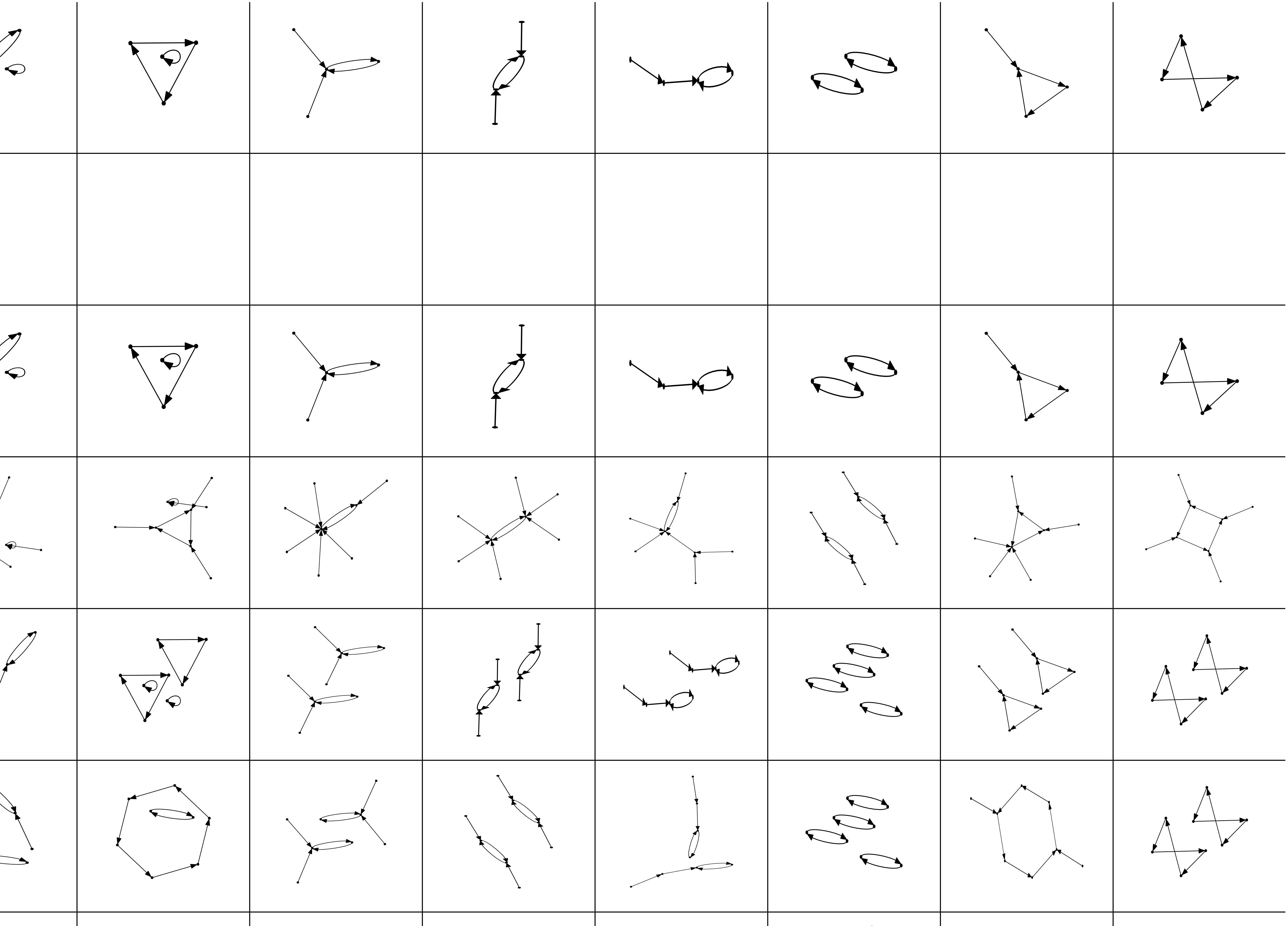
We forget the state names once again

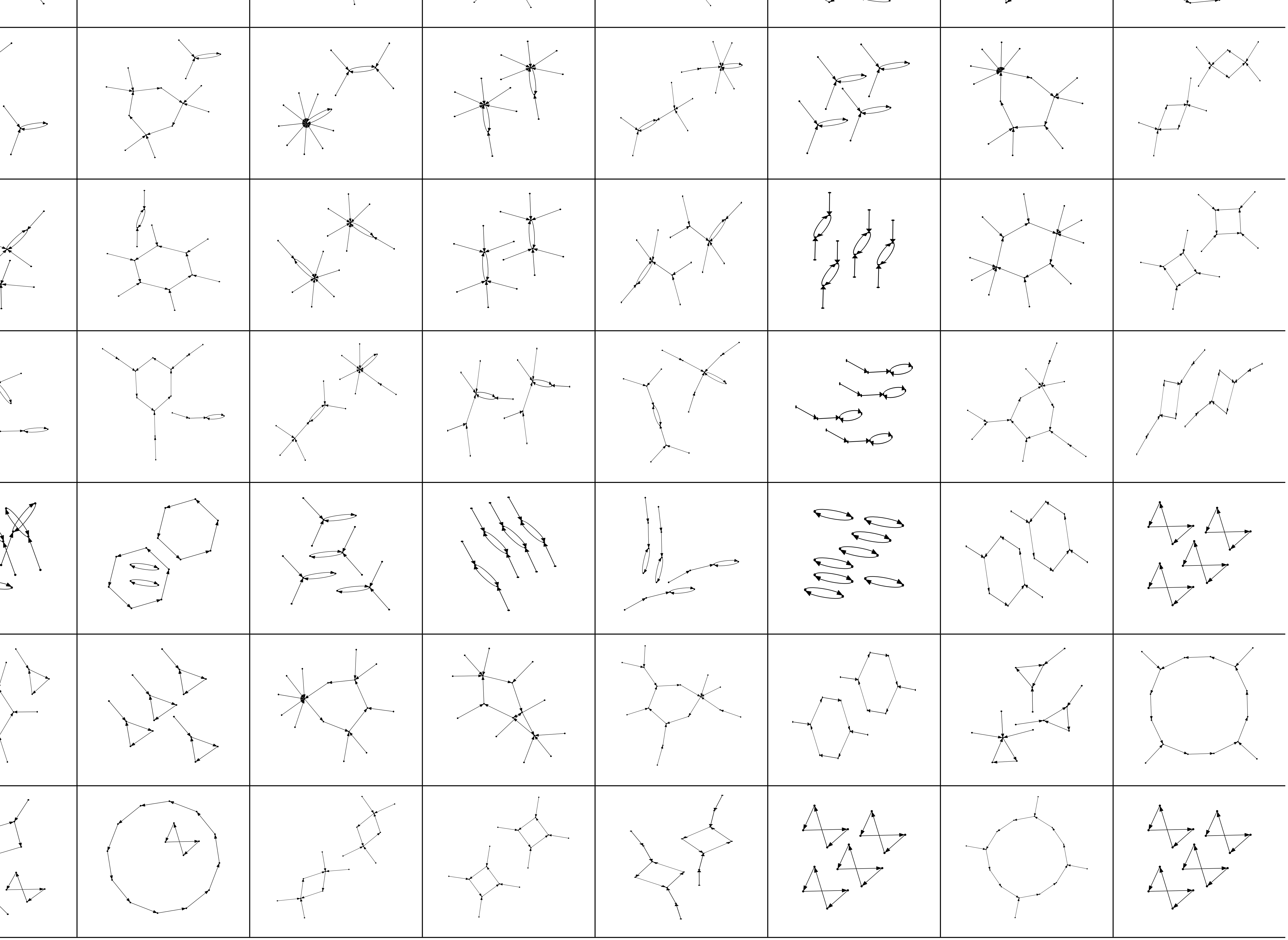


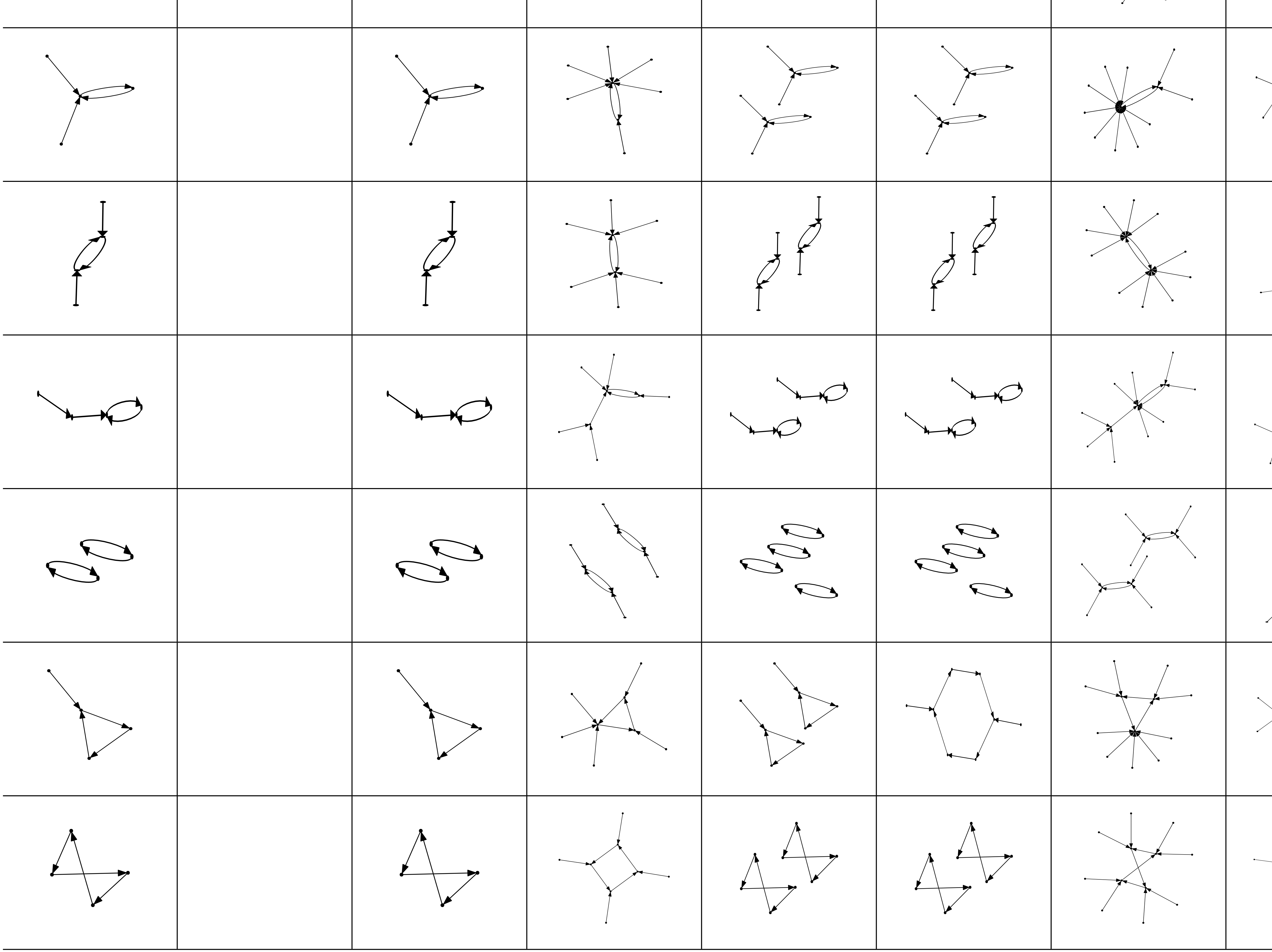
Introducing: the multiplication table, poster-size



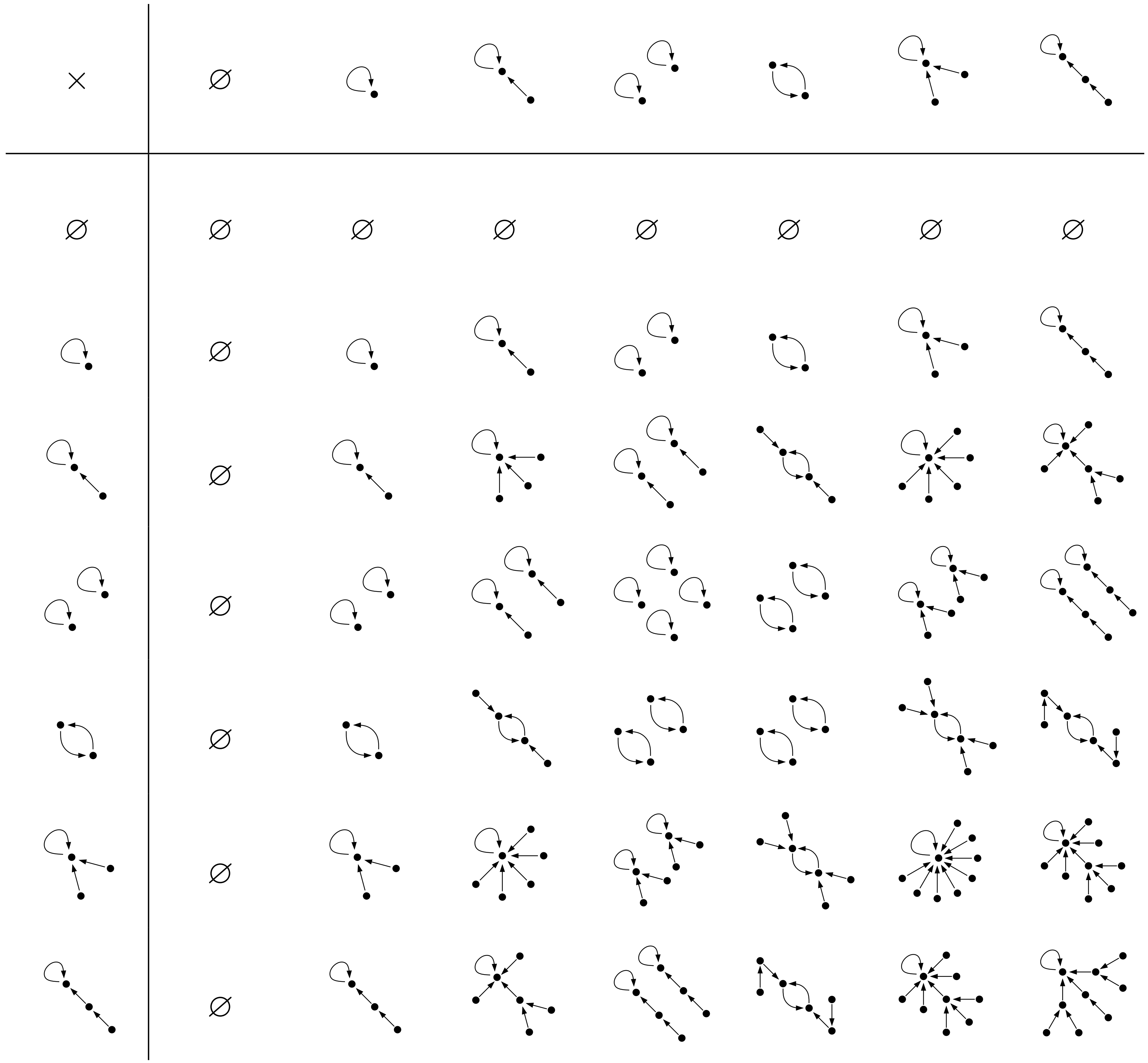








Prettier version



The semiring $\mathbf{D}$ of dynamical systems

$\mathbf{D}$ (modulo isomorphisms) is a semiring

Like a ring, without subtraction

- Product is (modulo isomorphism) commutative, associative and has identity $\mathbf{1} = (\{0\}, \mathrm{id})$; so, it's a commutative monoid
- Sum is (modulo isomorphism) commutative, associative and has identity $\mathbf{0} = (\varnothing, \varnothing)$; so, another commutative monoid
- The sum is the free commutative monoid (i.e., the multisets) over the set of connected, nonempty dynamical systems
- We also have a distributive law and the product annihilation law

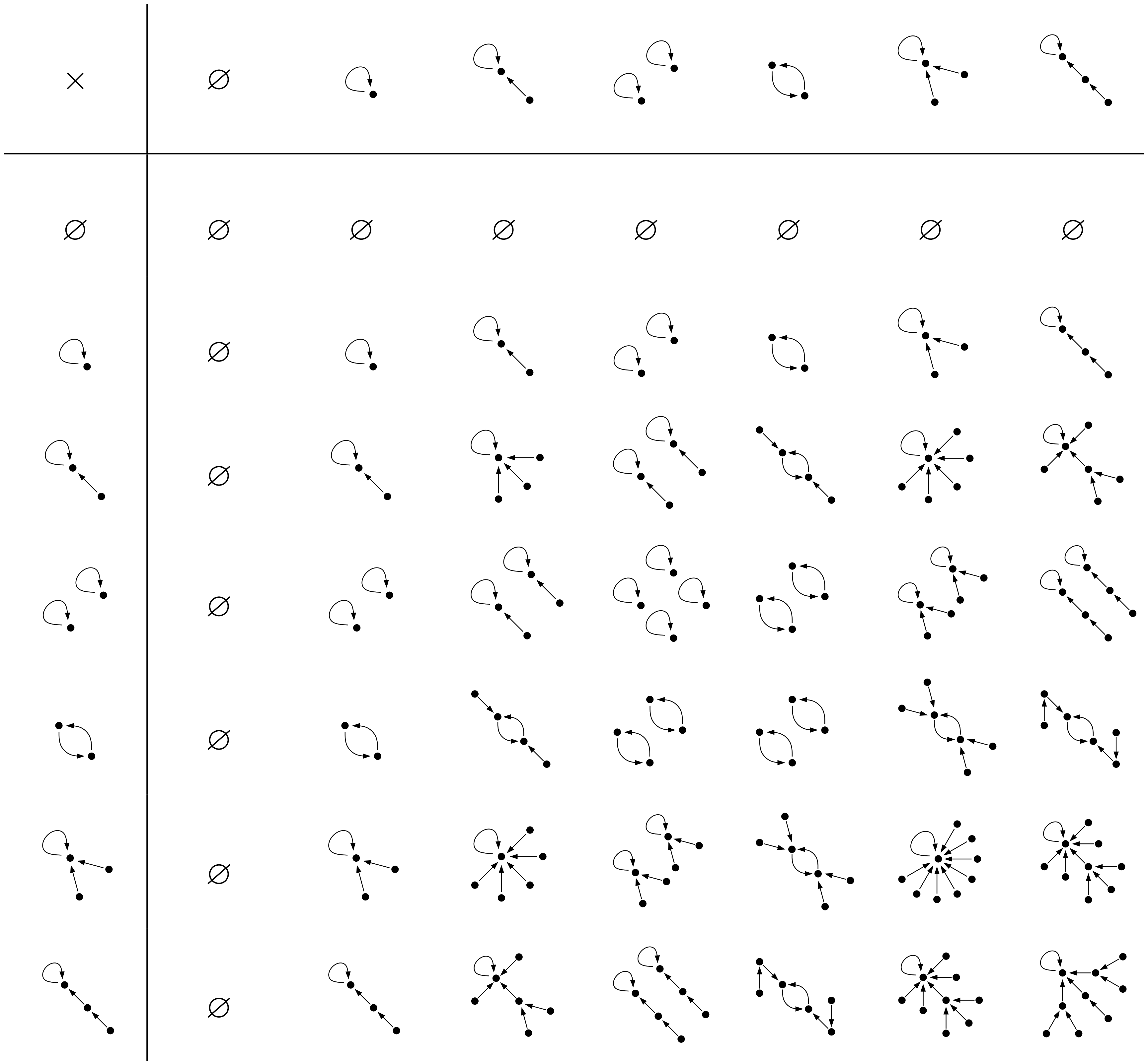
Motivation

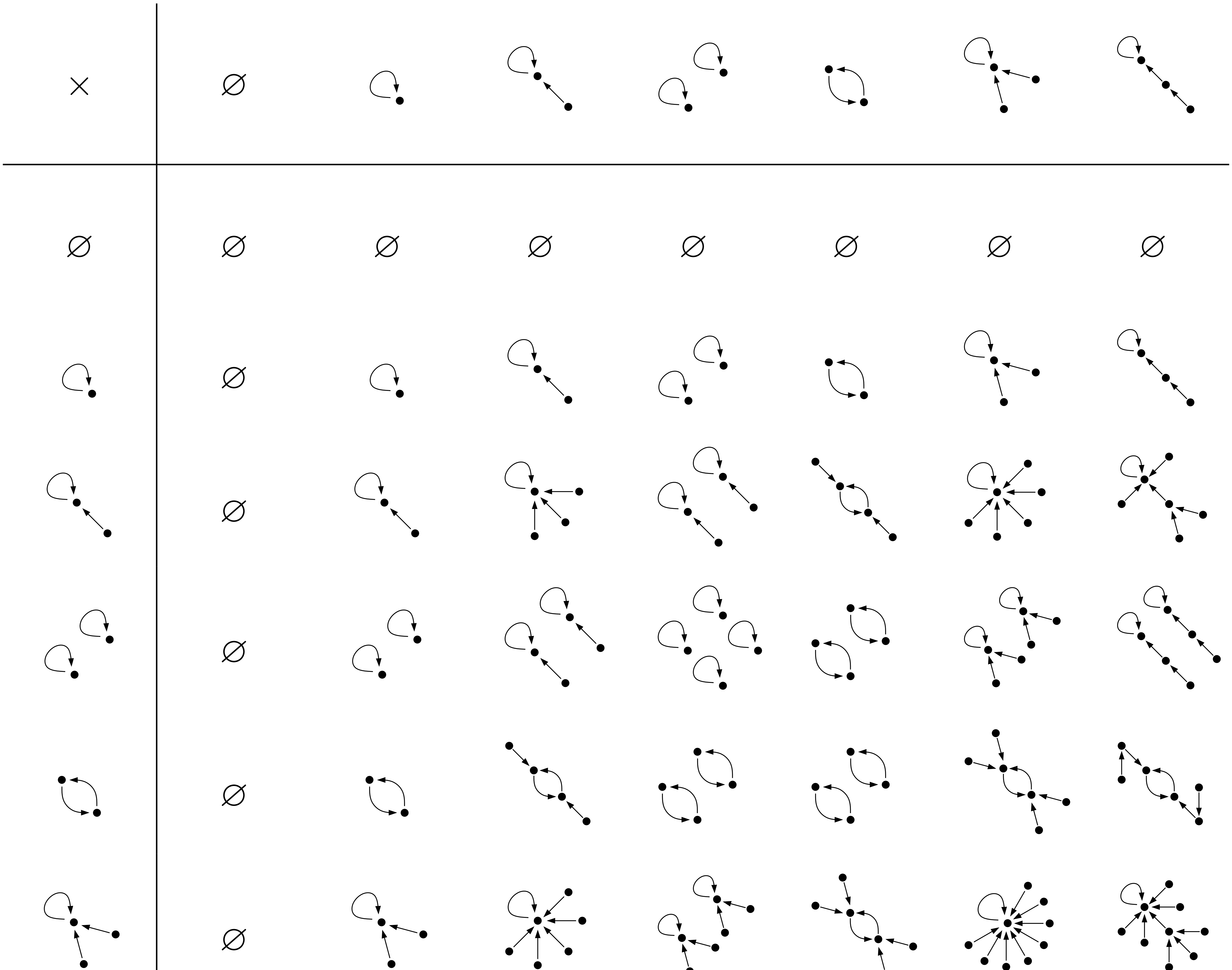
Exploit algebra to decompose large systems

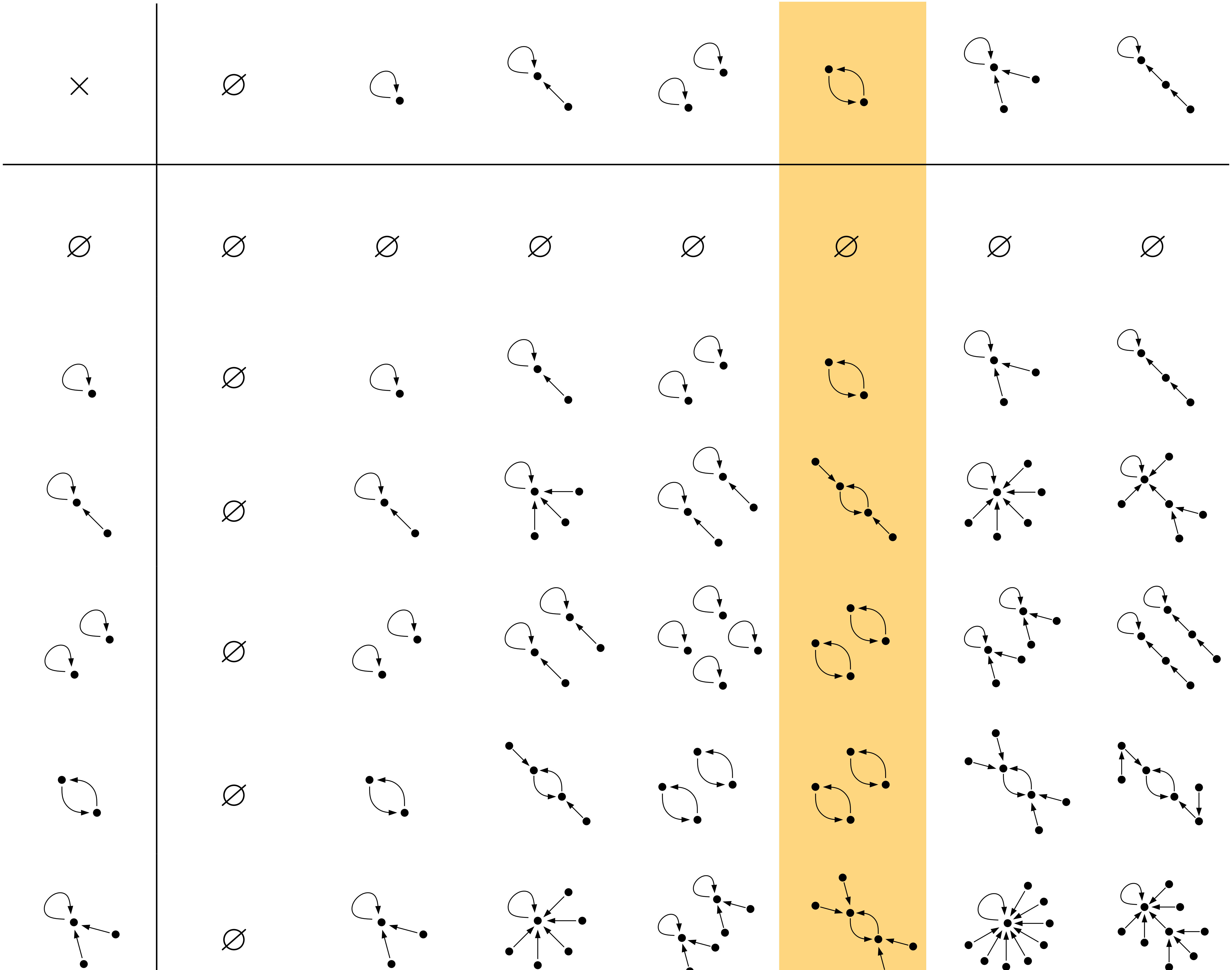
- Many decision problems on dynamical systems are intractable
- We try to make them “less intractable” by reducing the size
- We can deduce certain dynamical properties of complex systems in terms of the dynamics of its components
  - The fixed points of $A + B$ are those of A and those of B
  - The fixed points of $A \times B$ are pairs of fixed points of A and B
  - We can also compute number and lengths of cycles this way

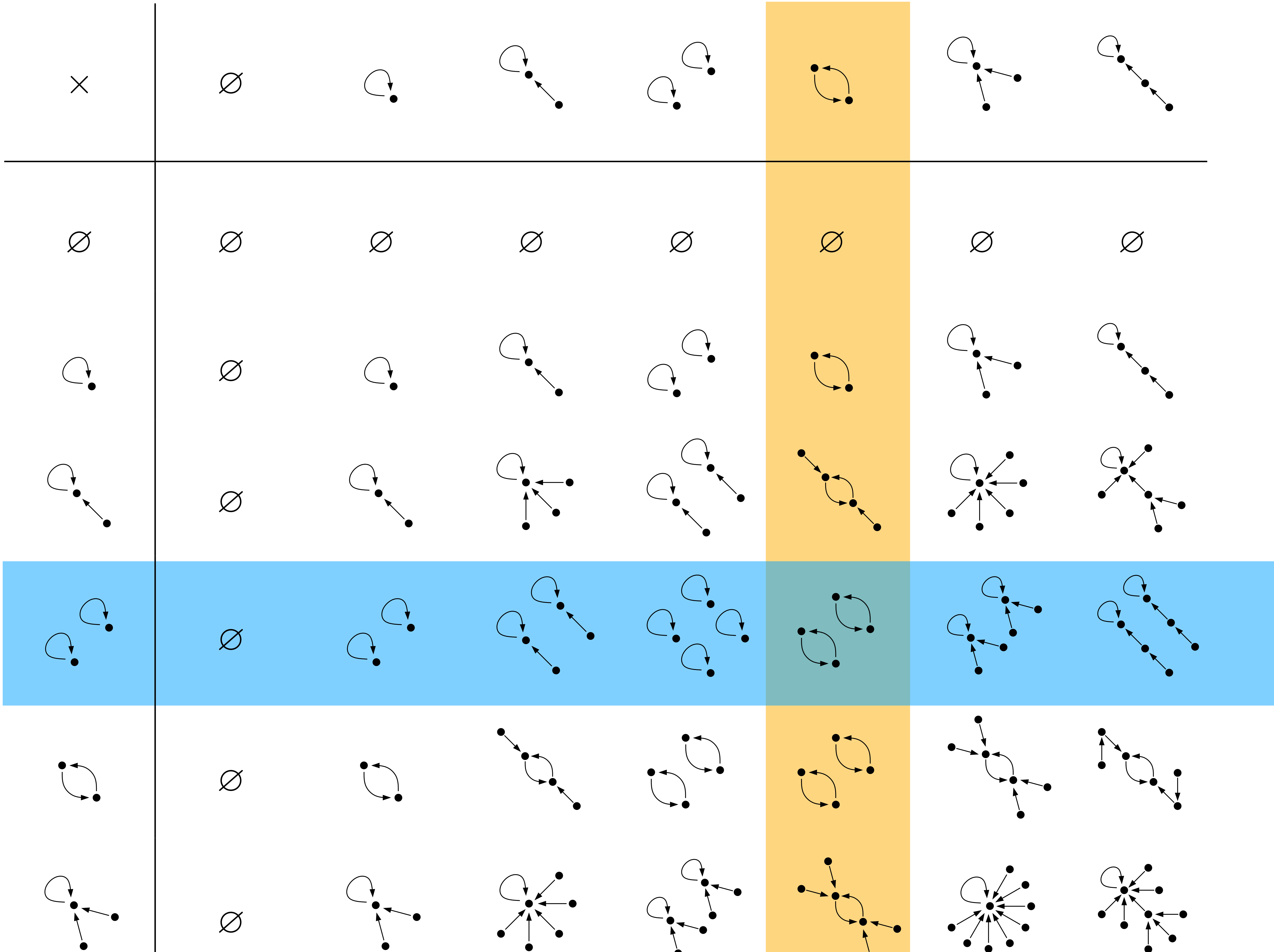
No unique factorisation!

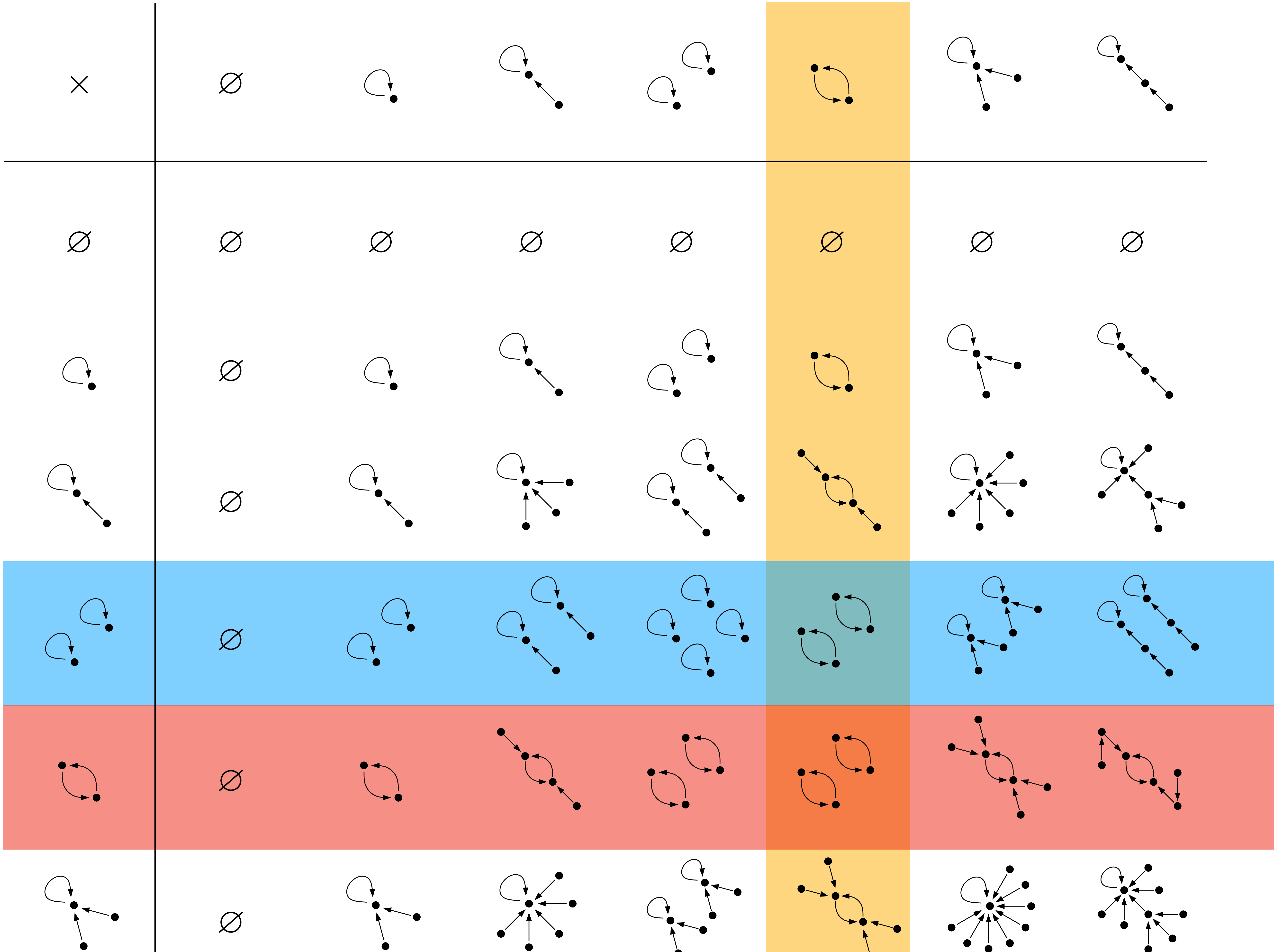
Multiplication table





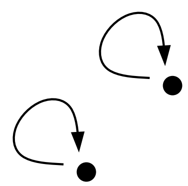
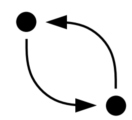
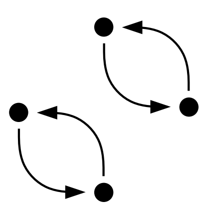


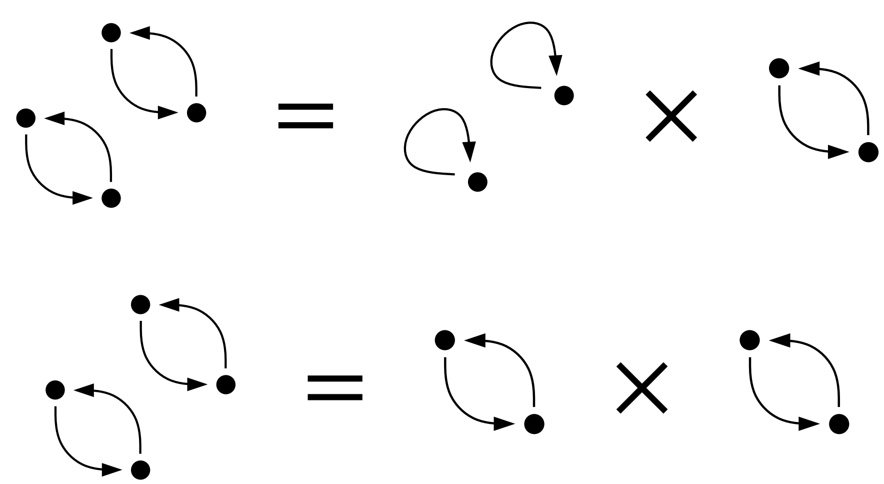




No unique factorisation

And the counterexample is minuscule

- The systems  and  are irreducible
- Any system with a prime number of states is irreducible, since the state space is a cartesian product
- So  has two distinct factorisations into irreducibles



Systems with arbitrarily many factorisations

Theorem

For each n, there exist a dynamical system with at least n factorisations

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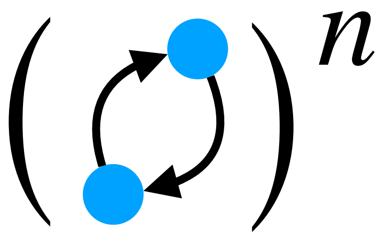
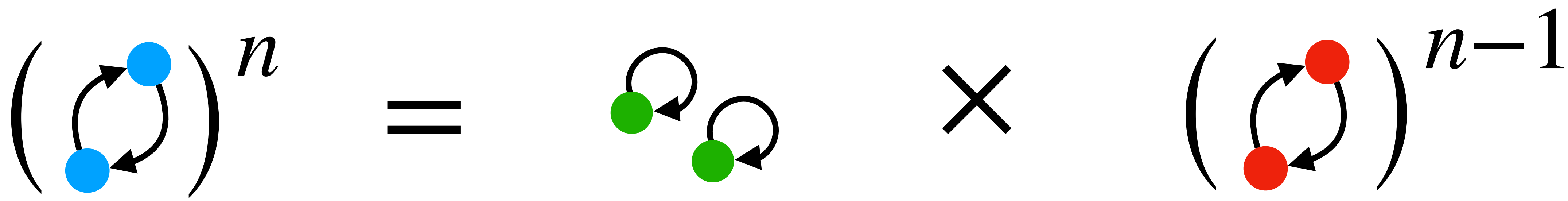


Theorem

For each n, there exist a dynamical system with at least n factorisations



Theorem

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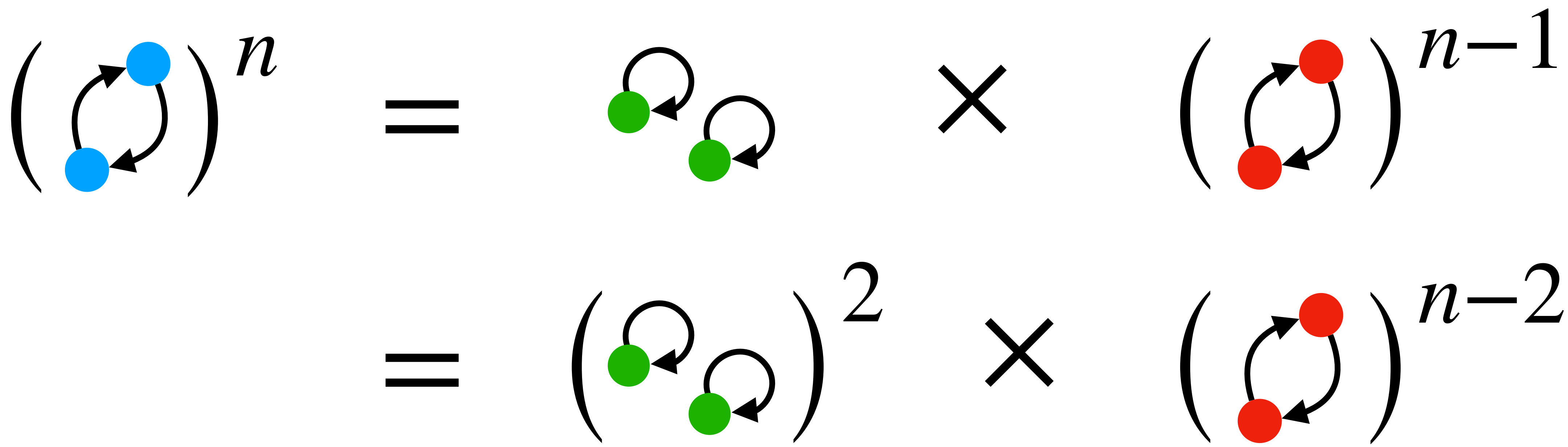

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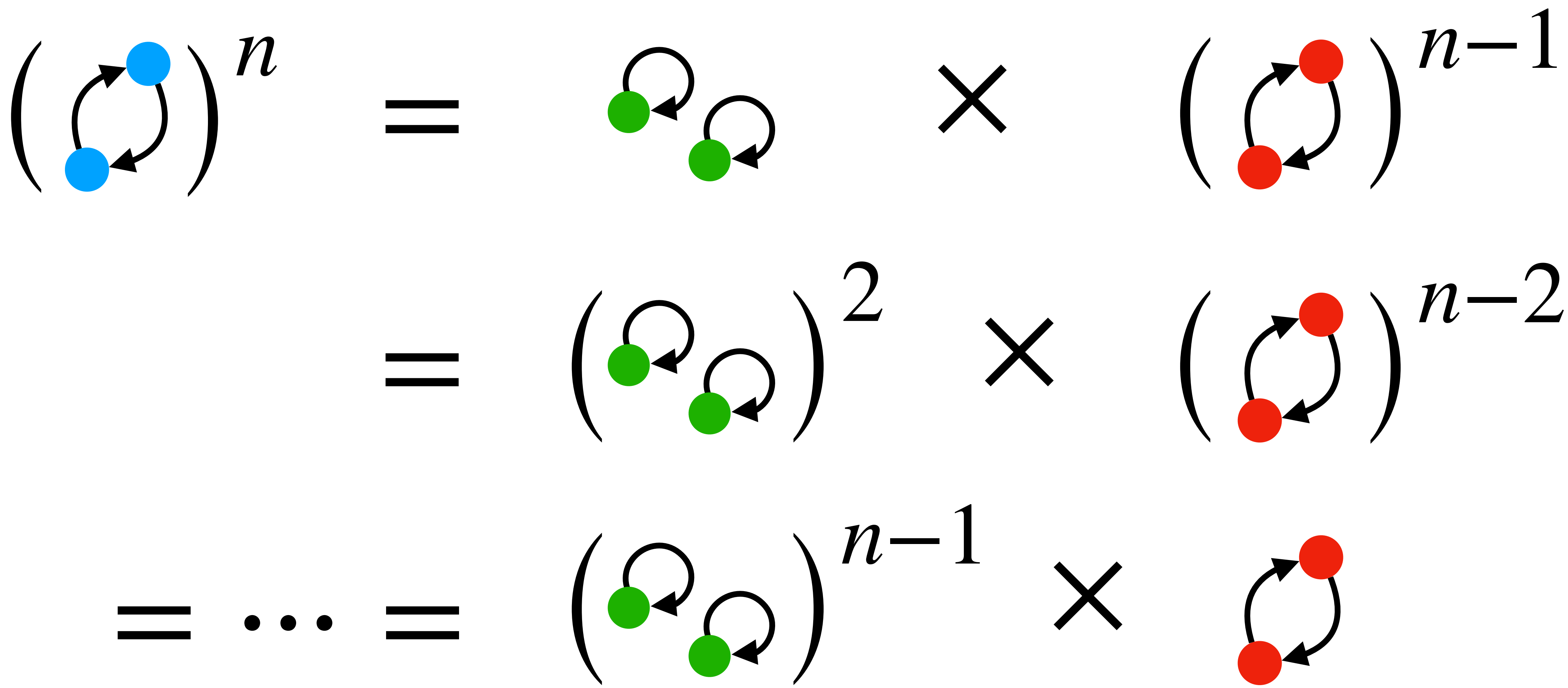


A notable subsemiring

$\mathbb{N}$ is a subsemiring of $\mathbf{D}$

This means trouble

- $\mathbb{N}$ is initial in the category of semirings
- Meaning that there is only one homomorphism $\varphi \colon \mathbb{N} \to \mathbf{D}$

$$\varphi(n) = \underbrace{\mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}}_{n \text{ times}} = \underbrace{\circlearrowleft\bullet + \circlearrowleft\bullet + \cdots + \circlearrowleft\bullet}_{n \text{ times}}$$

- In the case of $\mathbf{D}$, the homomorphism is injective, since $(\mathbf{D}, +)$ is the free monoid over connected, nonempty dynamical systems
- So $\mathbf{D}$ contains a isomorphic copy of $\mathbb{N}$

A bit more algebra, of the linear kind

$\mathbf{D}$ is a $\mathbb{N}$-semimodule

Like a vector space, but over a semiring

- Here the vectors are dynamical systems and the scalars are naturals
- Trivial because the semimodule axioms are a consequence of $\mathbb{N}$ being a subsemiring of $\mathbf{D}$:

$$n(A + B) = nA + nB \qquad (m + n)A = mA + nA$$

$$(mn)A = m(nA) \qquad 1A = A \qquad 0A = n\mathbf{0} = \mathbf{0}$$

- $\mathbf{D}$ as a semimodule has a unique, countably infinite basis consisting of all nonempty, connected dynamical systems
- The fact that $\mathbf{D}$ is a semimodule will be useful later

Irreducible systems

Most dynamical systems are irreducible

A is irreducible iff $A = BC$ implies $B = \mathbf{1}$ or $C = \mathbf{1}$

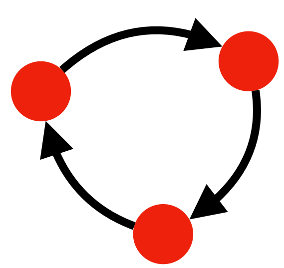
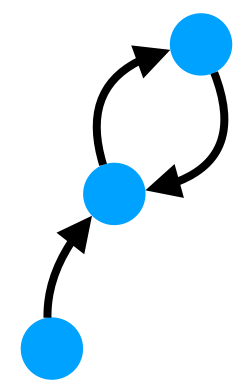
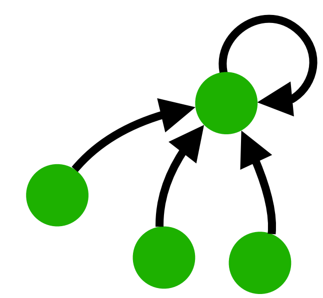
- Formally: $$\lim_{n\to\infty} \frac{\text{number of reducible systems over} \leq n \text{ states}}{\text{total number of systems over} \leq n \text{ states}} = 0$$
- The total number of systems over exactly n states is asymptotically $\eta\frac{\alpha^n}{\sqrt{n}}$, with $\eta \approx 0.443$ and $\alpha \approx 2.956$
- A reducible system over n states is the product of two systems with p and q states such that $pq = n$
- With a few summations and upper bounds, we get the result
- Notice that this is the opposite of the subsemiring $\mathbb{N}$

Polynomial equations over $\mathbf{D}[X_1, \ldots, X_m]$

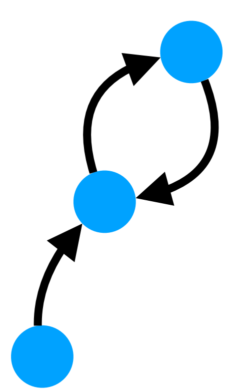
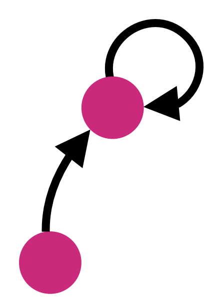
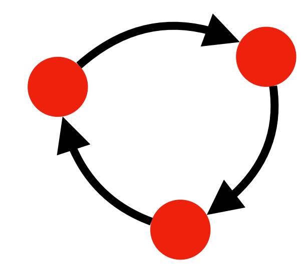
Polynomial equations over $\mathbf{D}[X_1, \ldots, X_m]$

For the analysis of complex systems

- Consider the equation

 $X + Y^2 =$  $Z +$ 

- There is least one solution

$X =$  $Y =$  $Z =$ 

Polynomial equations in semirings

As opposed to rings

- A ring has additive inverses (aka, it has subtraction)
- Each polynomial equation in a ring can be written as $p(\overrightarrow{X}) = 0$
- This is not the case for our semiring, which has no subtraction
- The general polynomial equation has the form $p(\overrightarrow{X}) = q(\overrightarrow{X})$ with two polynomials $p, q \in \mathbf{D}[\overrightarrow{X}]$

Solvability of polynomial equations over $\mathbf{D}$ is undecidable

Undecidability of polynomial equations

The spectre of Hilbert's 10th problem is haunting $\mathbf{D}$

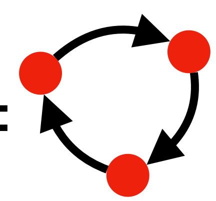
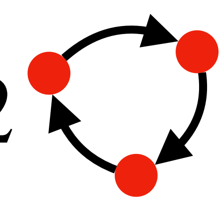
- We have showed that $\mathbb{N}$ is a subsemiring of $\mathbf{D}$
- But sometimes enlarging the solution space makes the problem actually easier: given $p, q \in \mathbb{N}[\overrightarrow{X}]$
  - Finding if $p(\overrightarrow{X}) = q(\overrightarrow{X})$ has solution in $\mathbb{N}$ is undecidable
  - Finding if $p(\overrightarrow{X}) = q(\overrightarrow{X})$ has solution in $\mathbb{R}$ is decidable
  - Finding if $p(\overrightarrow{X}) = q(\overrightarrow{X})$ has solution in $\mathbb{C}$ is trivial
- So, what about finding solutions in $\mathbf{D}$?

Natural polynomial equations

With non-natural solutions

- Let $p(X, Y) = 2X^2$ and $q(X, Y) = 3Y$ with $p, q \in \mathbb{N}[X, Y] \leq \mathbf{D}[X, Y]$

- Then $2X^2 = 3Y$ has the non-natural solution

$X =$  $\qquad Y = 2$ 

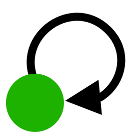
- But, of course, it also has the natural solution $X' = 3$, $Y' = 6$

- Notice how $X' = |X|$ and $Y' = |Y|$

- This is not a coincidence!

The function "size" $|\cdot|: \mathbf{D} \to \mathbb{N}$

It's a semiring homomorphism

- $|\varnothing| = 0$
- $|$  $| = 1$
- Since $+$ is the disjoint union, we have

$$|A + B| = |A| + |B|$$

- Since $\times$ is the cartesian product, we have

$$|AB| = |A| \times |B|$$

Notation for polynomials $p \in \mathbf{D}[\overrightarrow{X}]$

Of degree $\leq d$ over the variables $\overrightarrow{X} = (X_1, \ldots, X_k)$

$$p = \sum_{\vec{i} \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{X}^{\vec{i}}$$

where $\overrightarrow{X}^{\vec{i}} = \prod_{j=1}^{k} X_j^{i_j}$

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where $$\overrightarrow{X}^{\vec{i}} = \prod_{j=1}^{k} X_j^{i_j}$$

for instance $(X, Y, Z)^{(2,4,3)} = X^2 Y^4 Z^3$

Theorem

Solvability of natural equations

- If a polynomial equation over $\mathbb{N}[X_1, \ldots, X_k]$ has a solution in $\mathbf{D}^k$, then it also has a solution in $\mathbb{N}^k$

- In the larger semiring $\mathbf{D}$ we may find extra solutions, but only if the equation is already solvable over the naturals

- Then, by reduction from Hilbert's 10th problem, we obtain the undecidability in $\mathbf{D}$ of equations over $\mathbb{N}[\overrightarrow{X}]$…

- …and thus of arbitrary equations over $\mathbf{D}[\overrightarrow{X}]$

Proof

Consider $p(\overrightarrow{X}) = q(\overrightarrow{X})$ **with** $p, q \in \mathbb{N}[\overrightarrow{X}]$

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{X}^{\vec{i}} = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \overrightarrow{X}^{\vec{i}}$$

Proof

Suppose that $\overrightarrow{A} \in \mathbf{D}^k$ is a solution

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \overrightarrow{A}^{\vec{i}} = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \overrightarrow{A}^{\vec{i}}$$

Proof

Apply the size function $|\cdot|$

$$\left|\sum_{i\in\{0,\ldots,d\}^k} a_{\vec{i}}\overrightarrow{A}^{\vec{i}}\right| = \left|\sum_{i\in\{0,\ldots,d\}^k} b_{\vec{i}}\overrightarrow{A}^{\vec{i}}\right|$$

Proof

The size function $|\cdot|$ is a homomorphism

$$\sum_{\vec{i}\in\{0,\ldots,d\}^k} \left| a_{\vec{i}} \overrightarrow{A}^{\vec{i}} \right| = \sum_{\vec{i}\in\{0,\ldots,d\}^k} \left| b_{\vec{i}} \overrightarrow{A}^{\vec{i}} \right|$$

Proof

The size function $|\cdot|$ is a homomorphism

$$\sum_{i \in \{0,\ldots,d\}^k} |a_{\vec{i}}| \, |\overrightarrow{A}^{\vec{i}}| = \sum_{i \in \{0,\ldots,d\}^k} |b_{\vec{i}}| \, |\overrightarrow{A}^{\vec{i}}|$$

Proof

The coefficients are natural

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \, \color{red}{|} \, \overrightarrow{A}^{\vec{i}} \, \color{red}{|} = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \, \color{red}{|} \, \overrightarrow{A}^{\vec{i}} \, \color{red}{|}$$

Proof

We have $\overrightarrow{A}^{\vec{i}} = \prod_{j=1}^{k} A_j^{i_j}$

$$\sum_{i\in\{0,\ldots,d\}^k} a_{\vec{i}} \left| \prod_{j=1}^{k} A_j^{i_j} \right| = \sum_{i\in\{0,\ldots,d\}^k} b_{\vec{i}} \left| \prod_{j=1}^{k} A_j^{i_j} \right|$$

Proof

The size function $|\cdot|$ is a homomorphism

$$\sum_{i \in \{0,\ldots,d\}^k} a_{\vec{i}} \prod_{j=1}^{k} |A_j^{i_j}| = \sum_{i \in \{0,\ldots,d\}^k} b_{\vec{i}} \prod_{j=1}^{k} |A_j^{i_j}|$$

Proof

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Proof

So $|\overrightarrow{A}| = (|A_1|, \ldots, |A_k|)$ **is also a solution, QED**

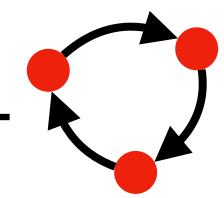
$$p(|A_1|, \ldots, |A_k|) = q(|A_1|, \ldots, |A_k|)$$

Equations with non-natural coefficients

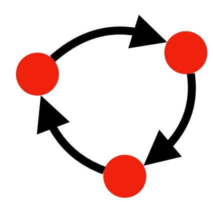
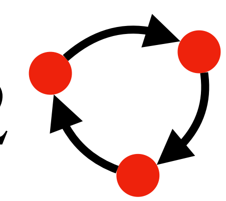
Equations without natural solutions

They do exist

- Consider, for instance

$$X^2 = Y +$$ 

- This equation has solution

$$X =$$  $$Y = 2$$ 

- But there is no natural solution, because the RHS is non-natural and cannot be made natural by adding stuff

Polynomial equations with constant RHS are decidable and in $\mathbf{NP}$

Nondeterministic algorithm

For $p(\overrightarrow{X}) = C$ with $C \in \mathbf{D}$

- Since $+$ and $\times$ are monotonic wrt the sizes of the operands, each X_i in a solution to the equation has size $\leq |C|$

- So it suffices to guess a dynamical system of size $\leq |C|$ for each variable in polynomial time, then calculate LHS

- Finally we check whether LHS and RHS are isomorphic, exploiting the fact that graph isomorphism is in $\mathbf{NP}$

- Only one caveat: if at any time during the calculations the LHS becomes larger than $|C|$, we halt and reject (otherwise the algorithm might take exponential time)

Systems of linear equations with constant RHS are $\mathbf{NP}$-complete

NP-hardness of linear systems

By reduction from One-in-three-3SAT

- Given a 3CNF Boolean formula φ, is there a satisfying assignment such that exactly one literal per clause is true?
- For each variable x of φ we have one equation $X + X' = 1$, forcing one between X and X' to be 1, and the other to be 0
- For each clause, for instance $(x \vee \neg y \vee z)$, we have one equation $X + Y' + Z = 1$, which forces exactly one variable to 1
- These are all linear, constant-RHS equations over $\mathbf{D}[\overrightarrow{X}]$ (actually $\mathbb{N}[\overrightarrow{X}]$), and its solutions are the same as the satisfying assignments of φ with one true literal per clause

A single linear,
constant-RHS equation
is $\mathbf{NP}$-complete

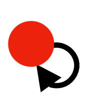
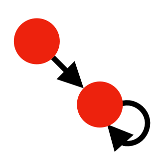
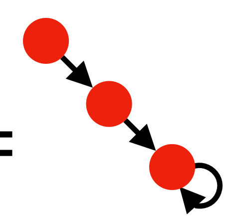
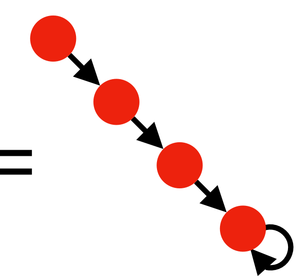
Reducing the system of equations to one

Several $\mathbb{N}[\overrightarrow{X}]$ linear equations to one $\mathbf{D}[\overrightarrow{X}]$ equation

- Let $p_1(\overrightarrow{X}) = 1, \ldots, p_n(\overrightarrow{X}) = 1$ be the previous system of equations, with $p_i \in \mathbb{N}[\overrightarrow{X}]$

- Recall that $\mathbf{D}$ is a $\mathbb{N}$-semimodule with basis all connected systems

- Take any n easy-to-compute, linearly independent systems $e_1, \ldots e_n \in \mathbf{D}$, for instance

$e_1 =$  $e_2 =$  $e_3 =$  $e_4 =$  $\cdots$

- Then the equation $e_1 p_1(\overrightarrow{X}) + \cdots + e_n p_n(\overrightarrow{X}) = e_1 + \cdots + e_n$ is a linear equation over $\mathbf{D}[\overrightarrow{X}]$ having the same solutions as the original system

Open problems

Open problems

Algebraic ones

- Are there prime elements P, that is, whenever P divides AB it divides either A or B? What do they represent?
    - We know exactly zero prime elements 🤷‍♂️
- Does it make any sense to adjoin the additive inverses in order to obtain a ring?
    - Think about imaginary numbers, using them in intermediary computation steps, but discarding any imaginary solutions
- Is it useful to find nondeterministic dynamical system (i.e., arbitrary graph) solutions to equations?
- Semirings of infinite discrete-time dynamical systems

Open problems

Computability and complexity

- Find larger classes of solvable equations, e.g., by number of variables or degree of the polynomials
  - Do we obtain the same results as for natural numbers?
- The semiring of computably infinite dynamical systems
- Discover classes of equations solvable efficiently
  - Hard for systems in succinct form
- Find out if there exist decidable equations harder than $\mathbf{NP}$
  - It would feel strange to jump from $\mathbf{NP}$ to undecidable

Bibliography 📖

Something to read before bed

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Thanks for your attention!
Any questions?