Analysis of discrete dynamical systems

An algebraic perspective

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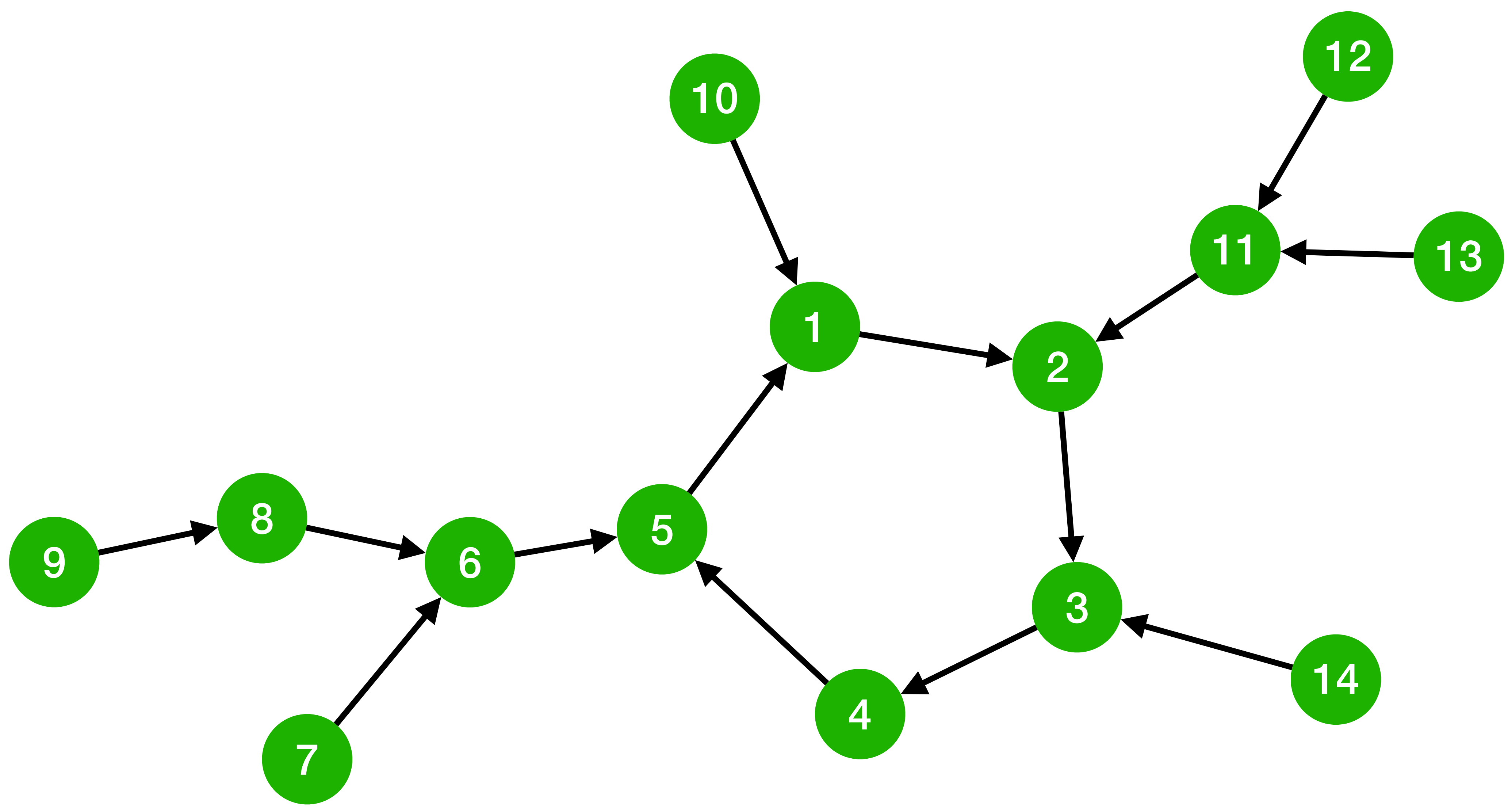
Discrete (finite, deterministic) dynamical systems



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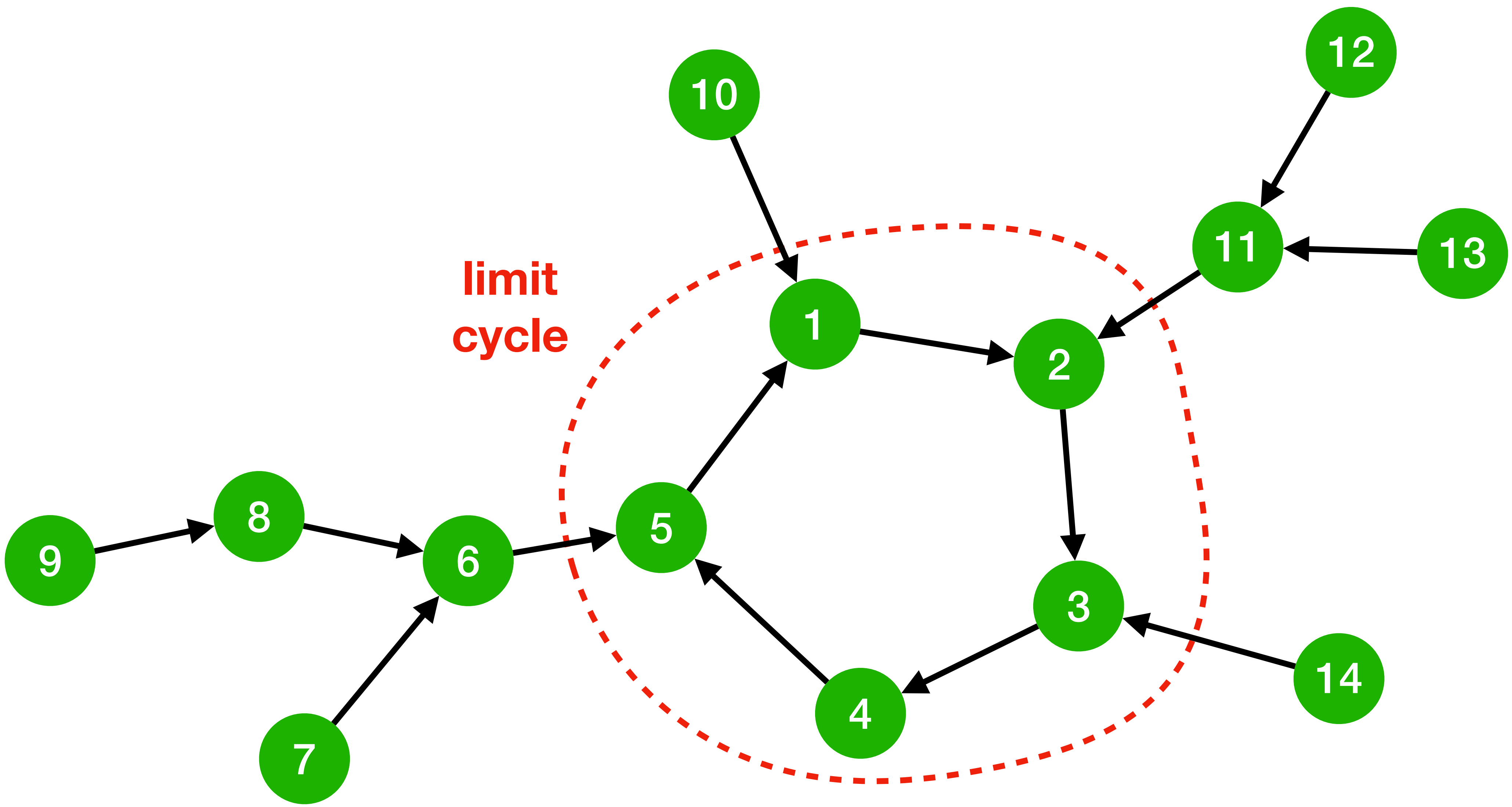

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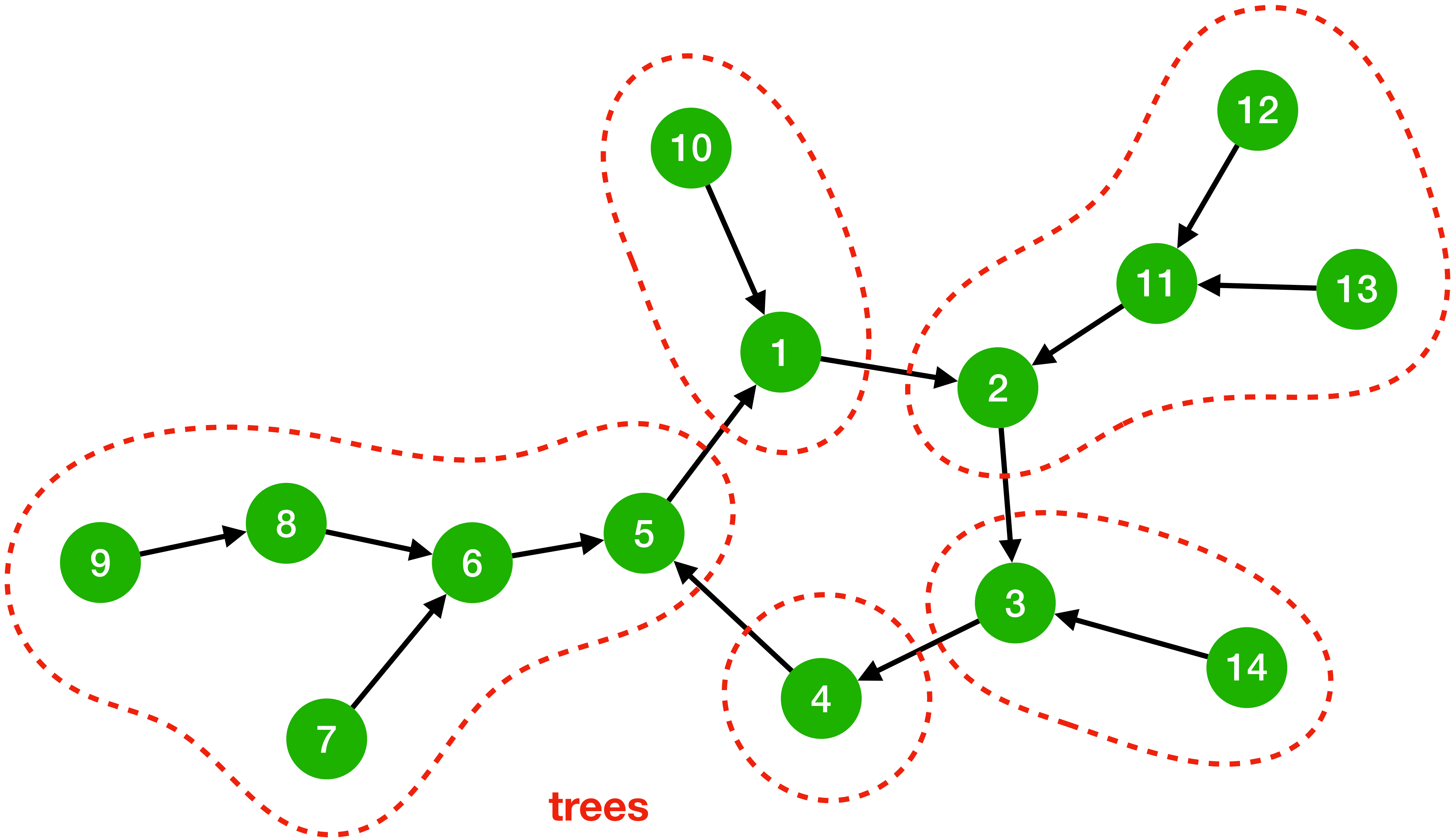
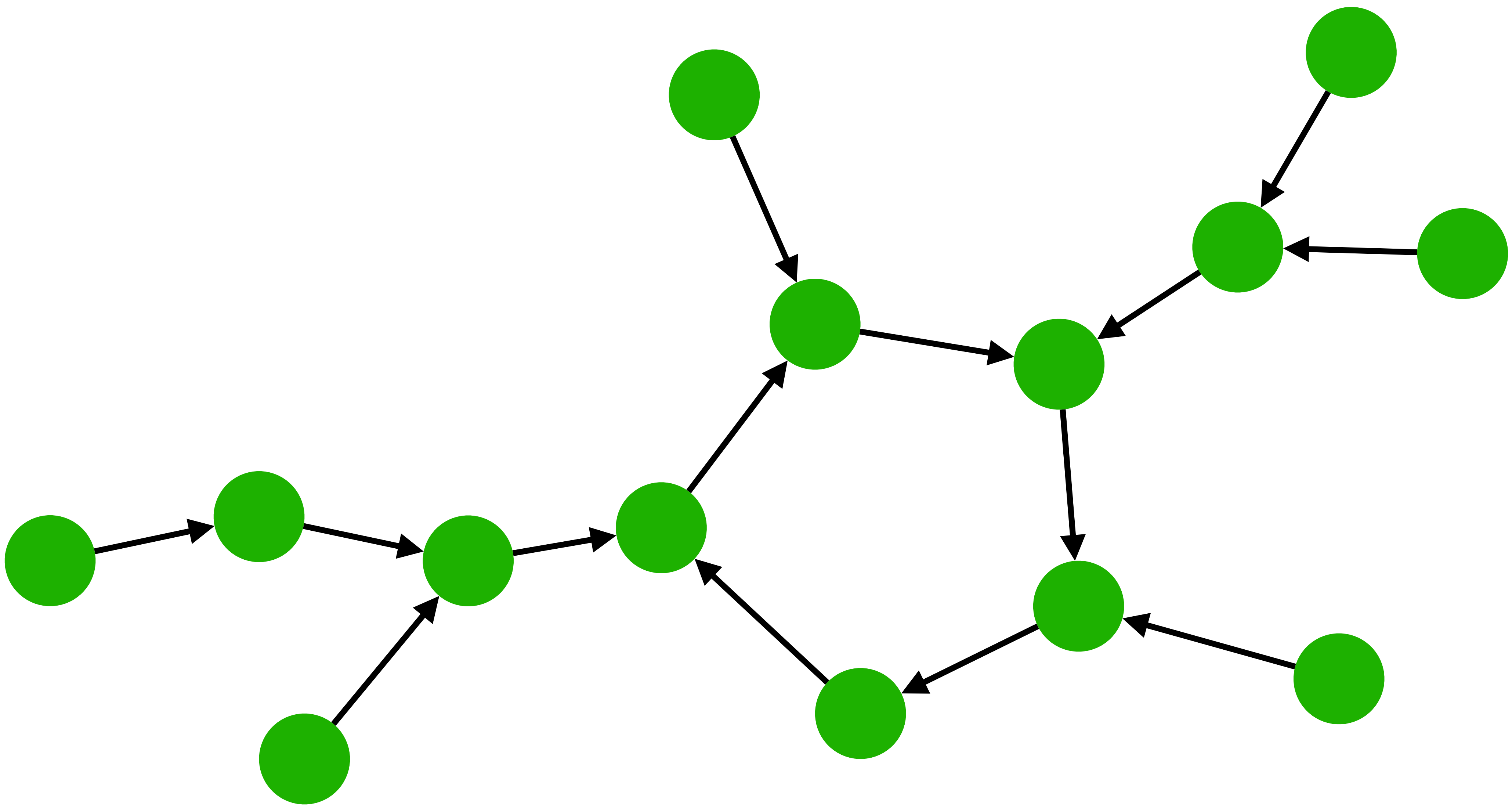


Discrete (finite, deterministic) dynamical systems up to isomorphisms



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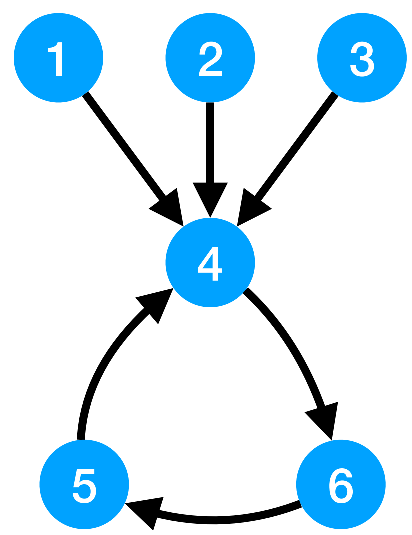

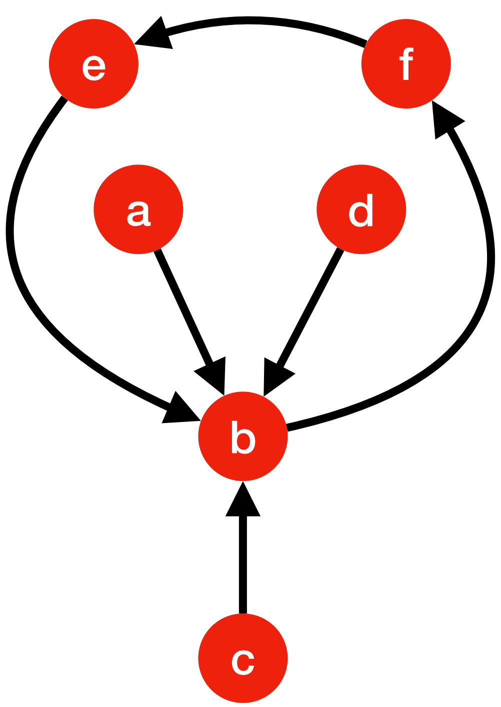

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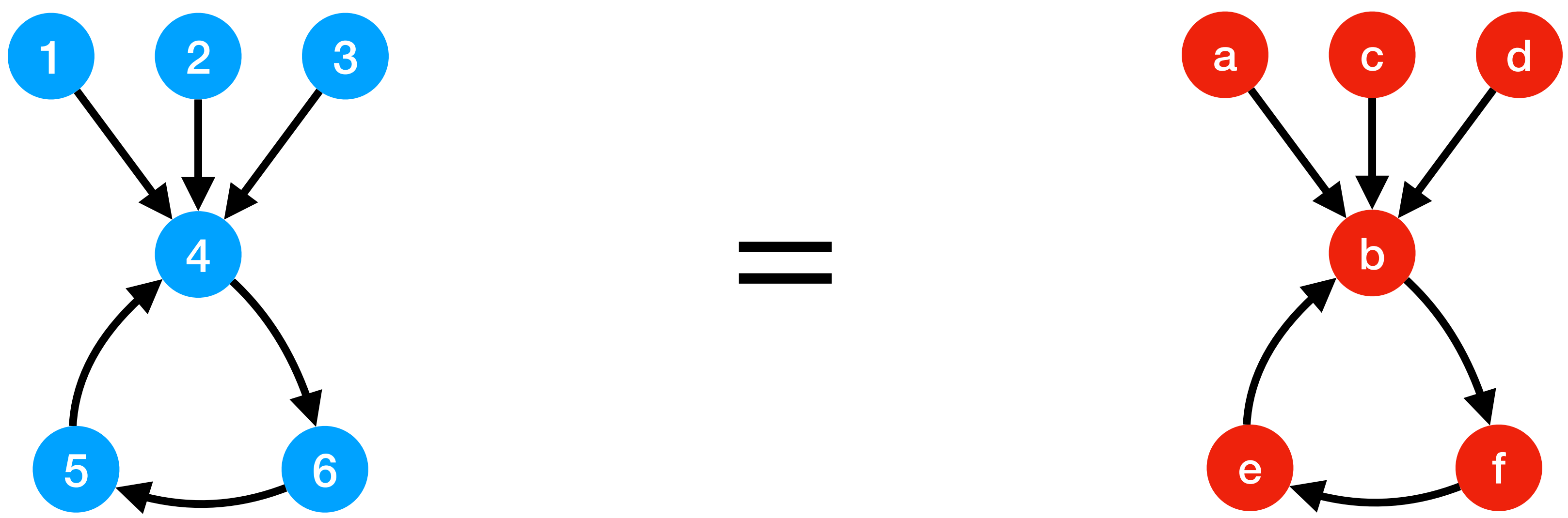
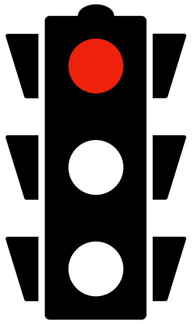


An example from engineering

Traffic lights



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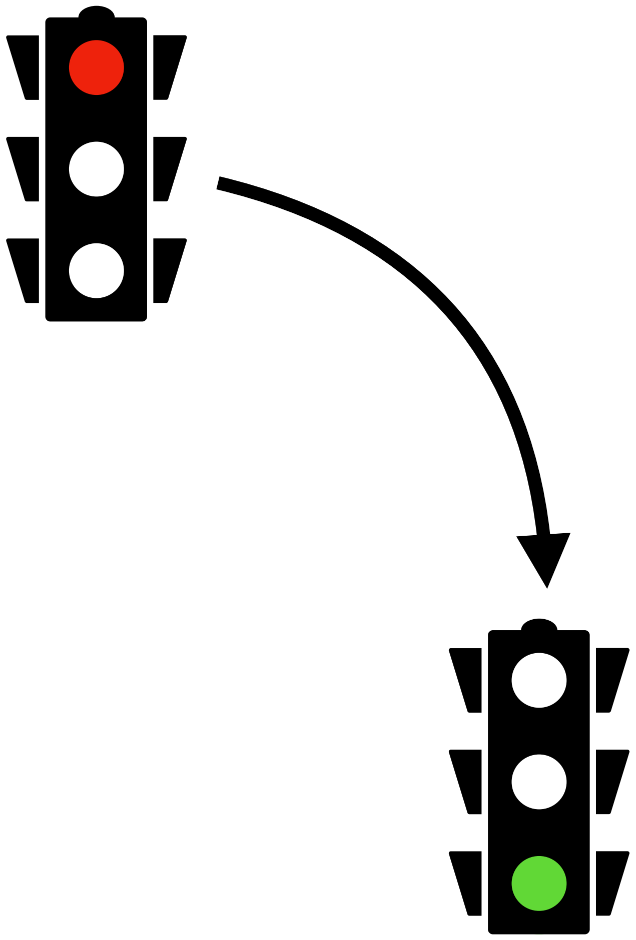

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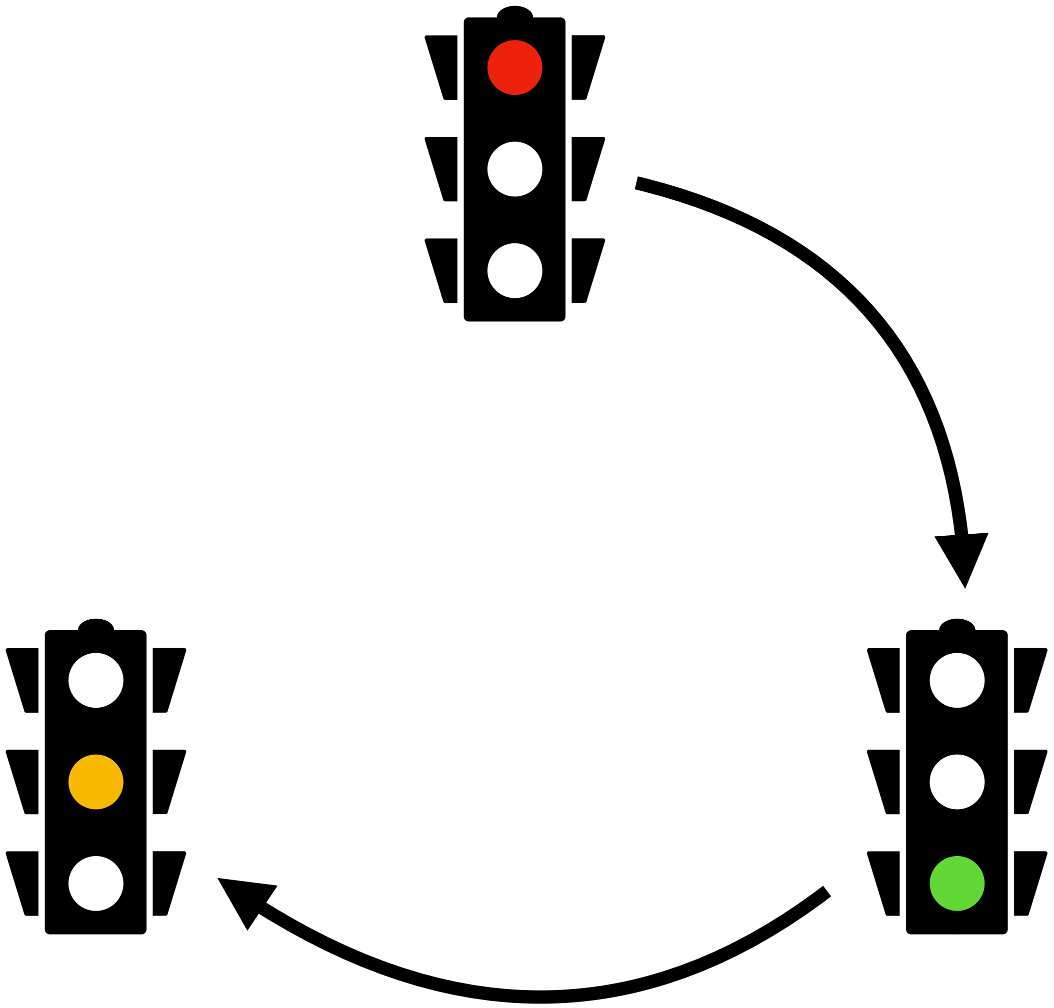

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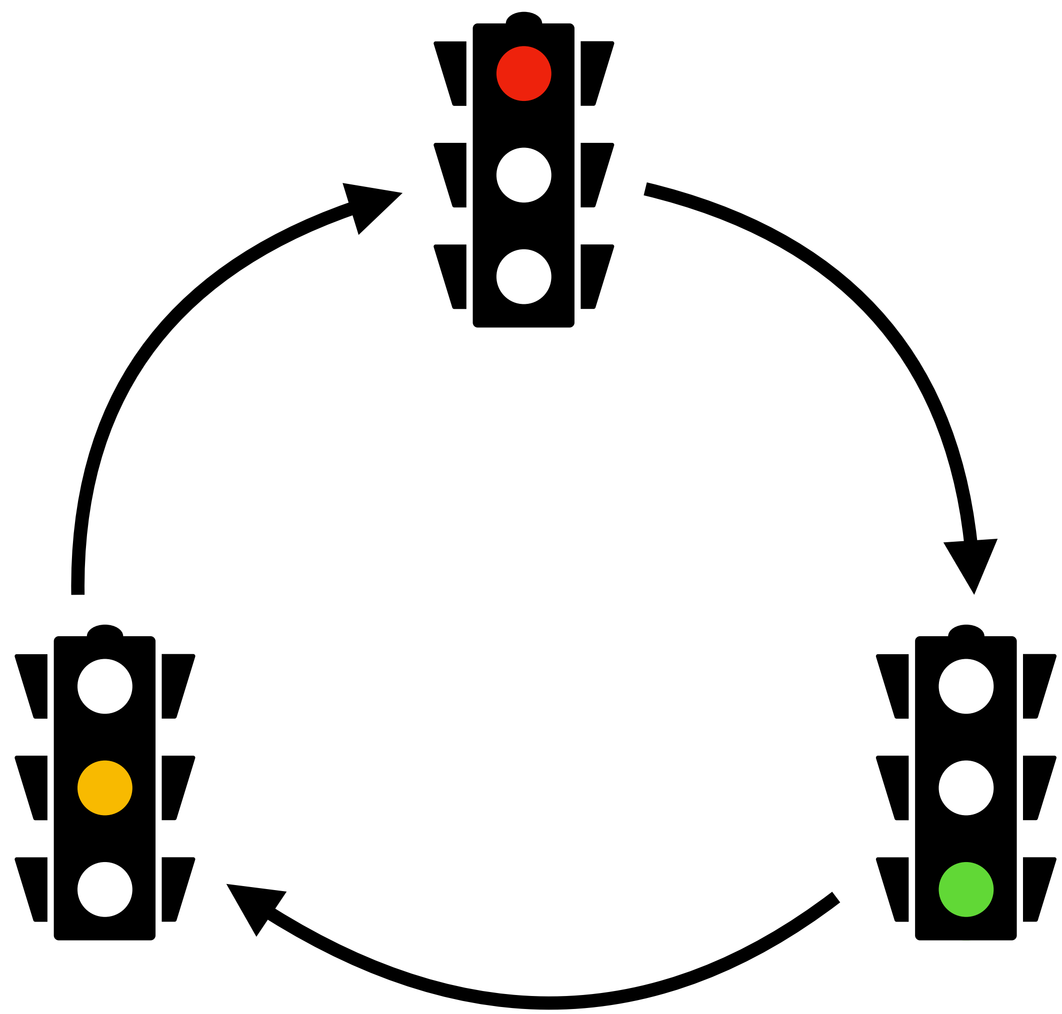

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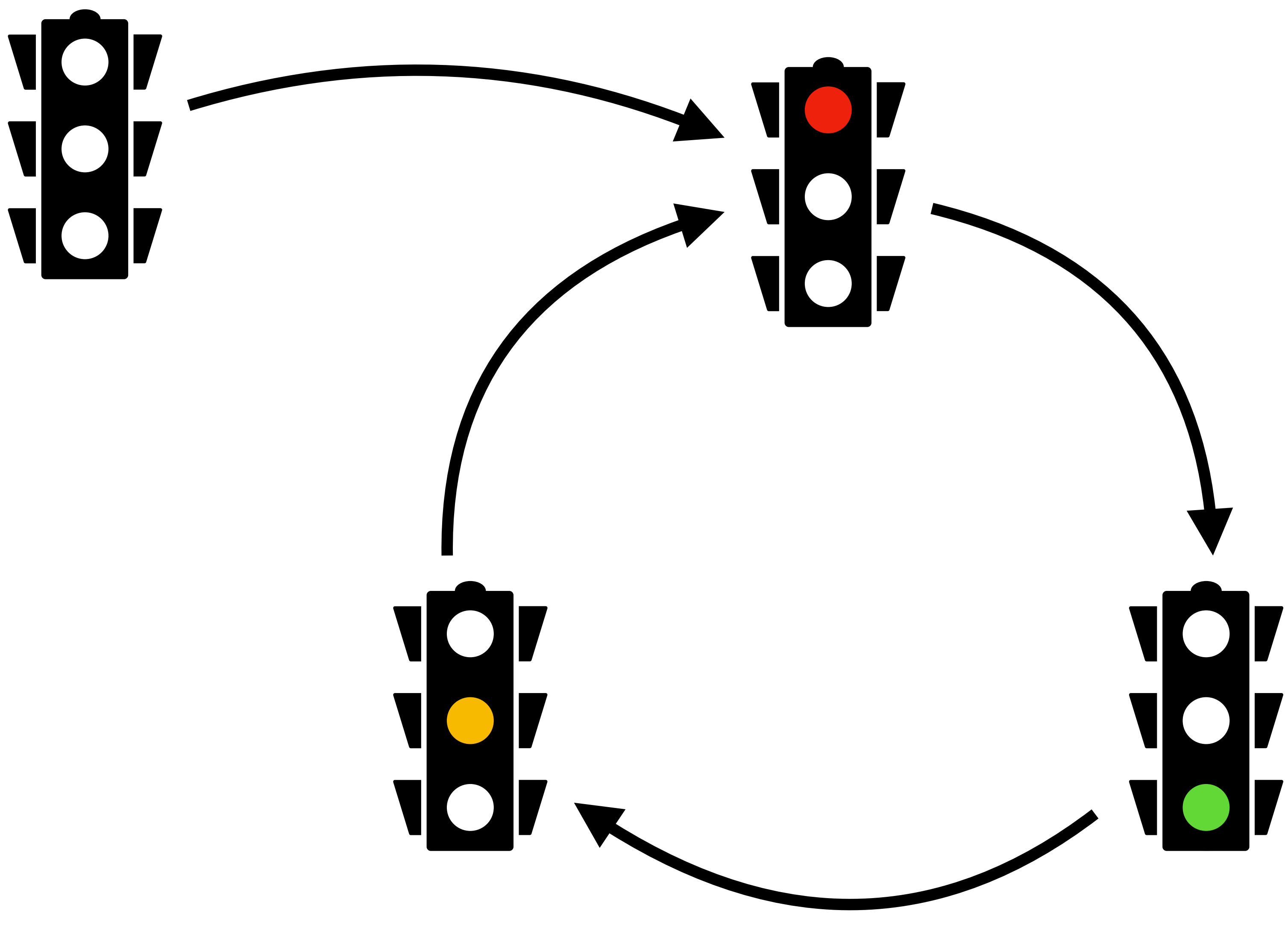
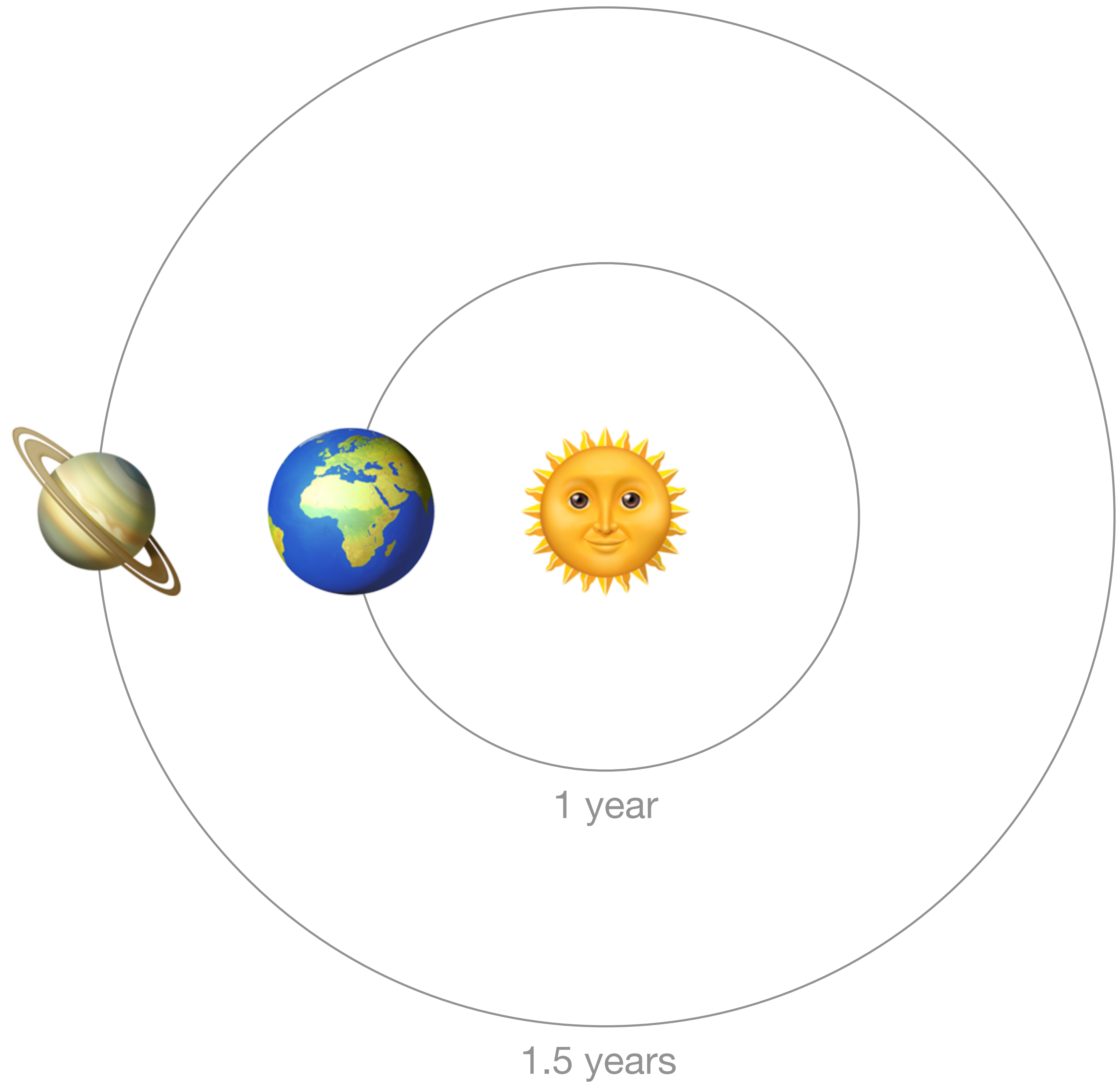
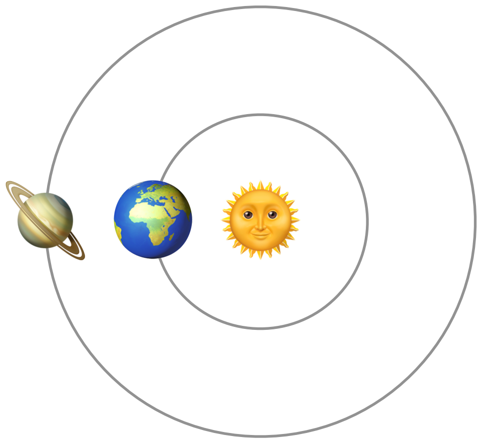


An example from science

A planetary system



Evolution in time



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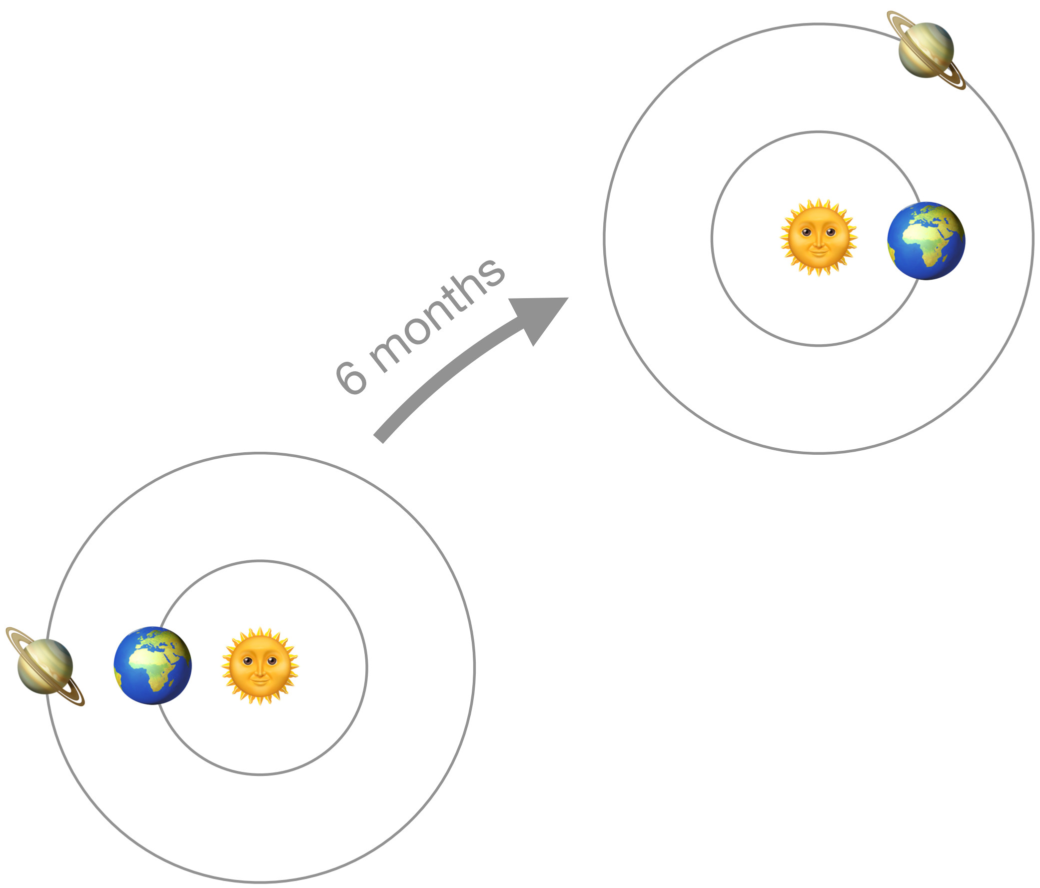

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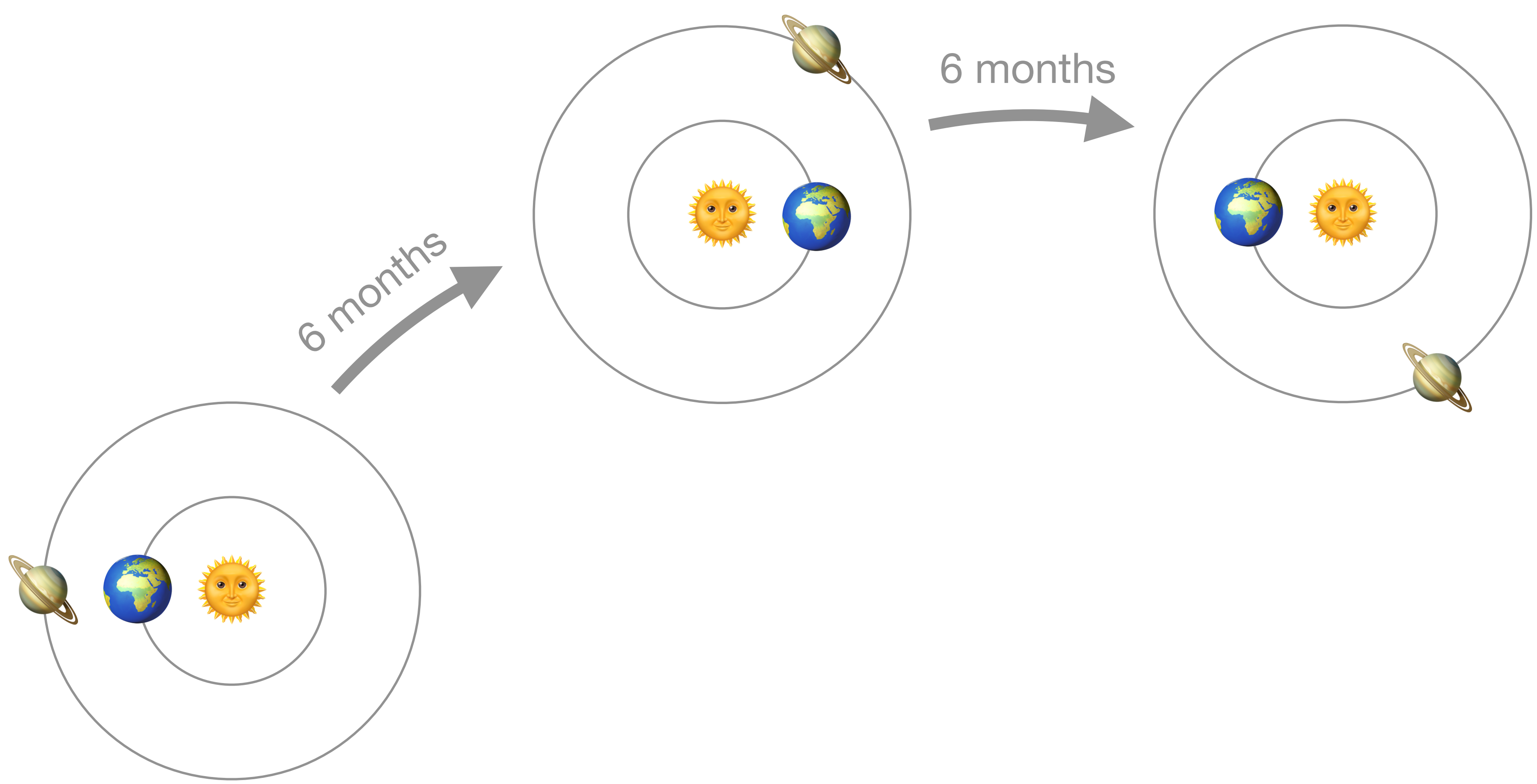

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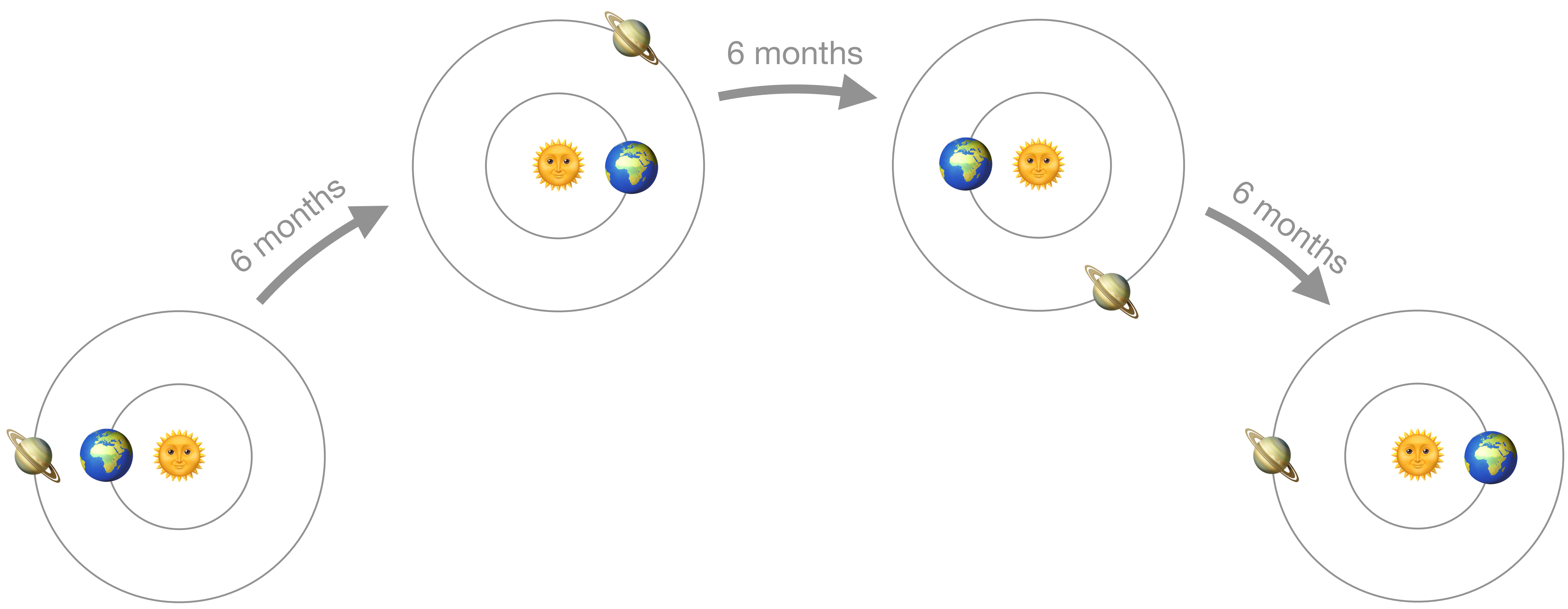

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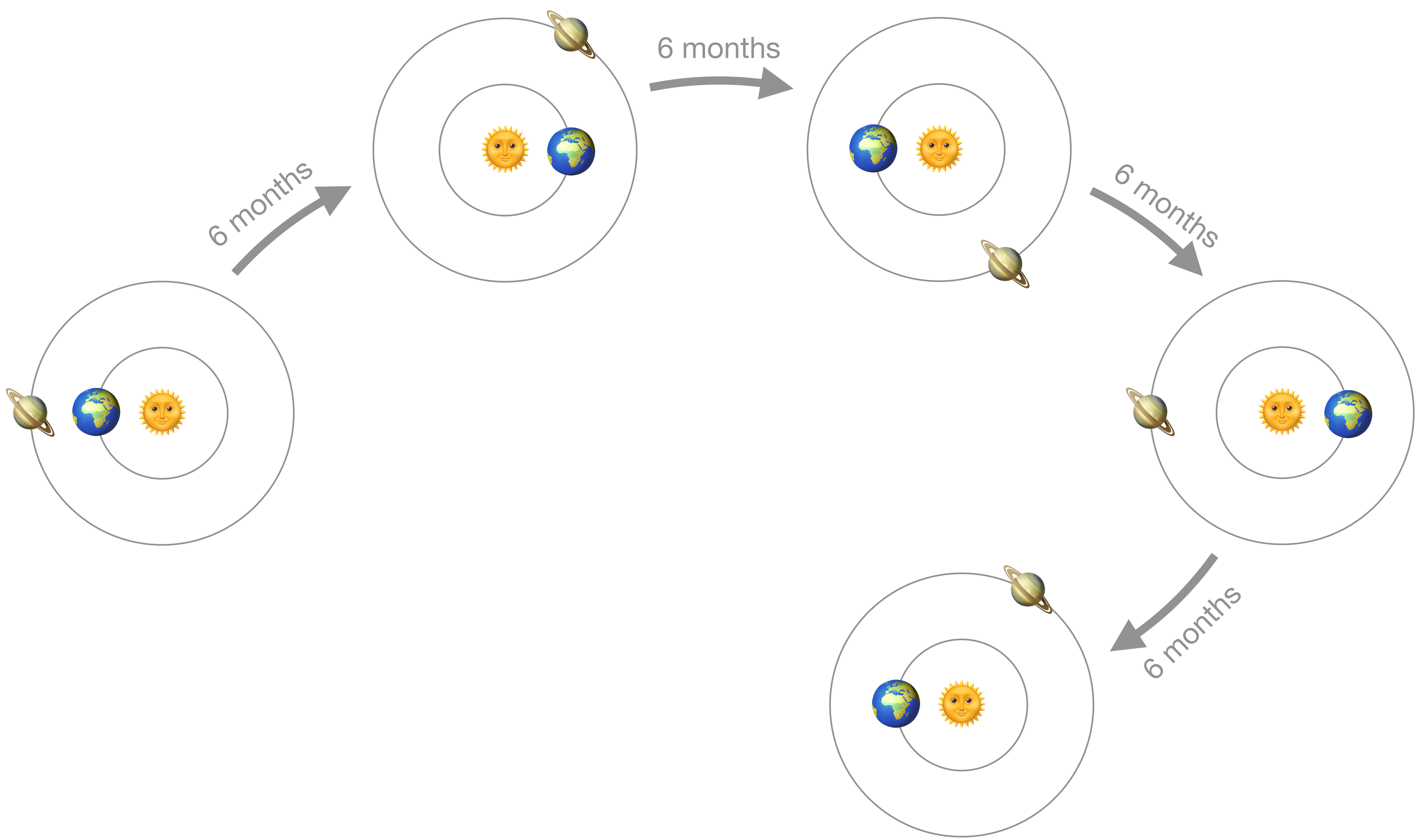

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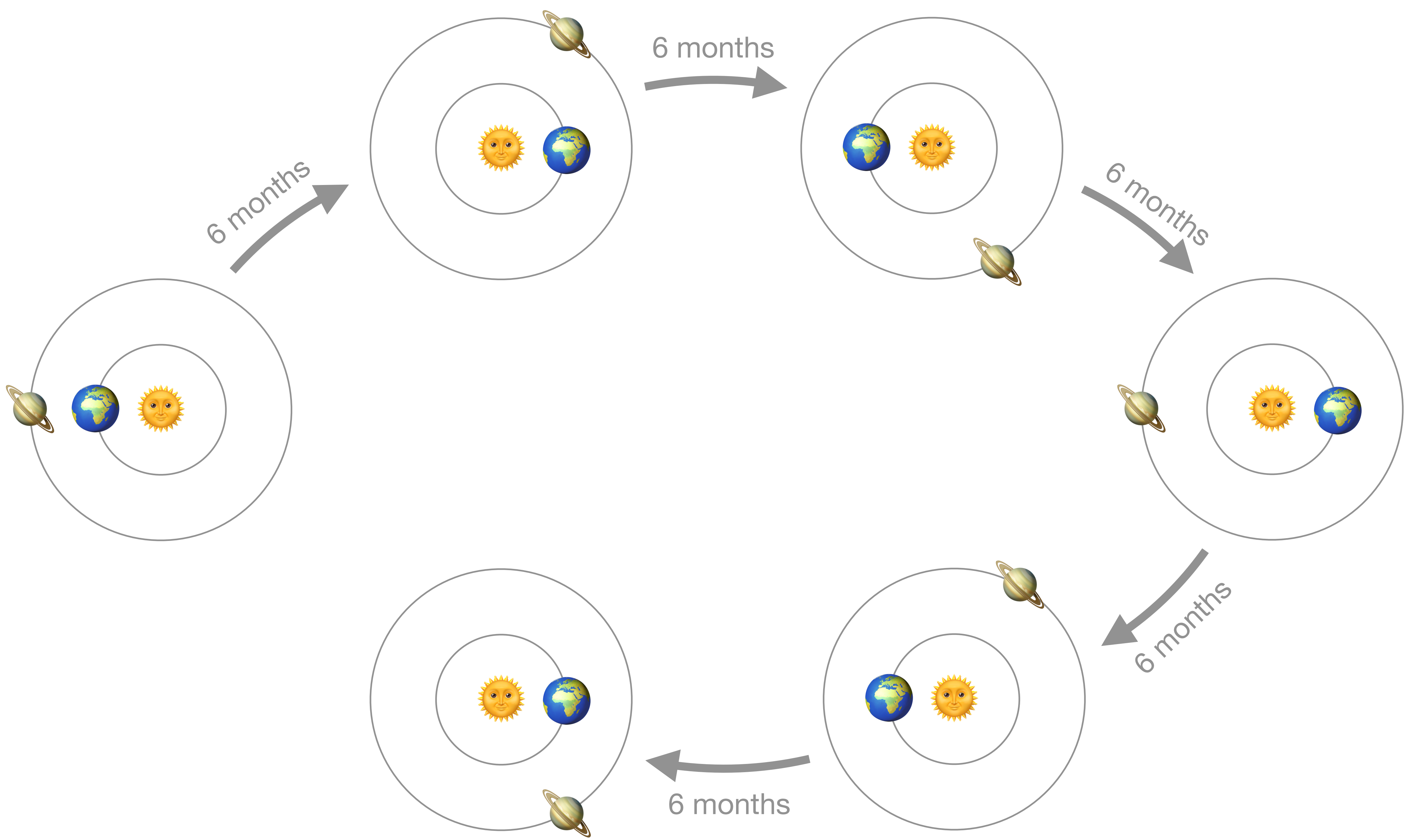

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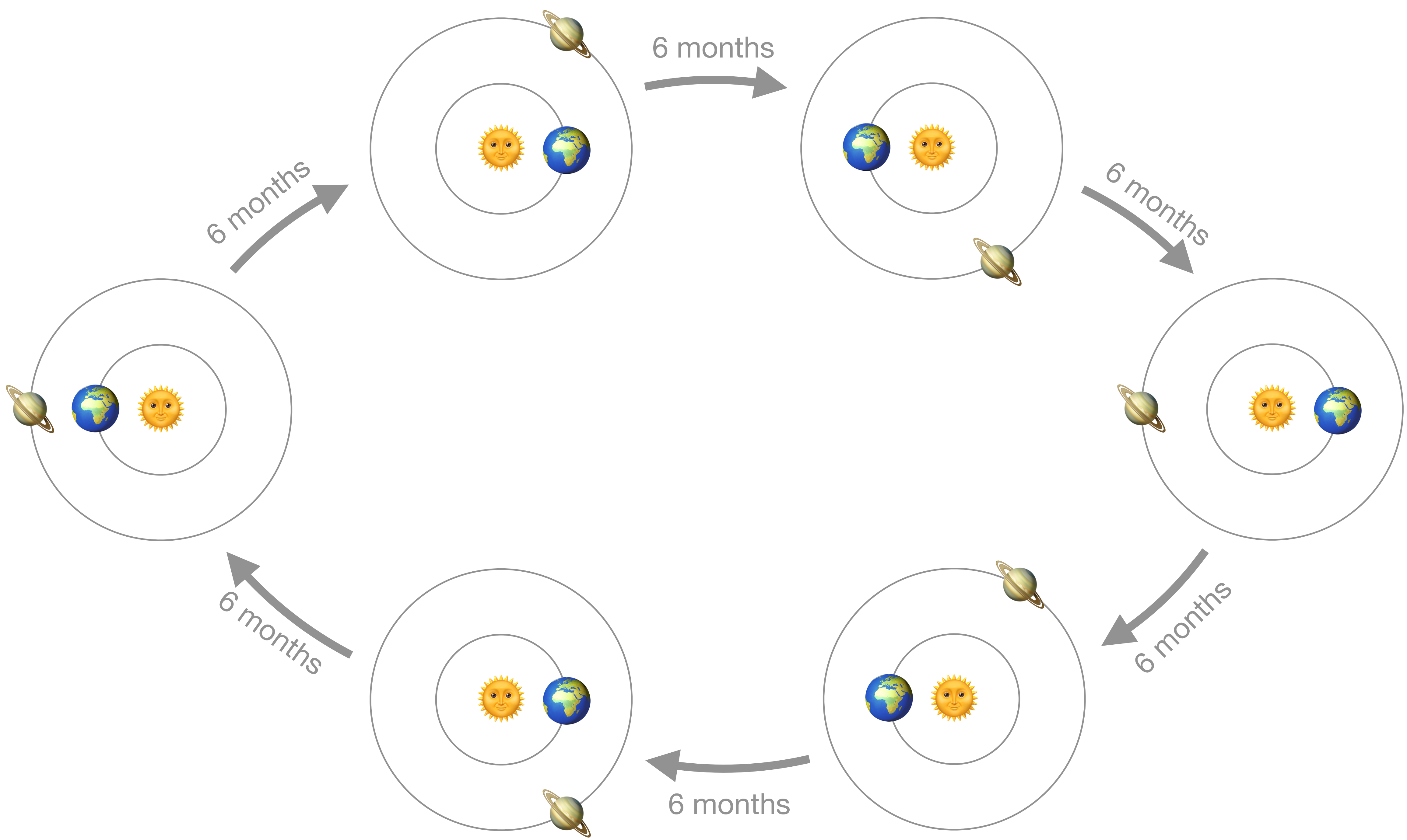

Decomposing the system

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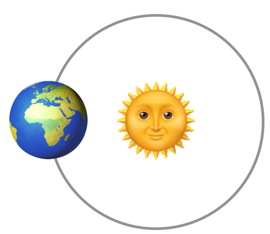

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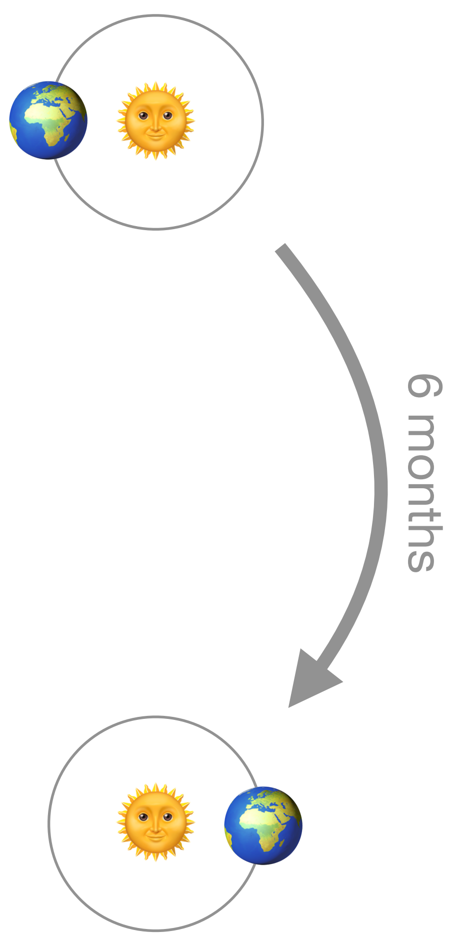

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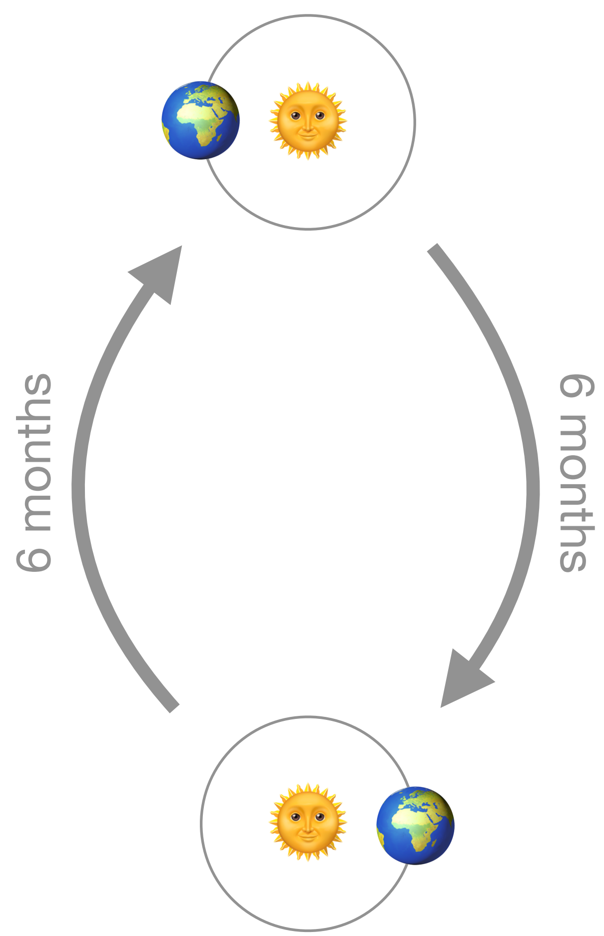

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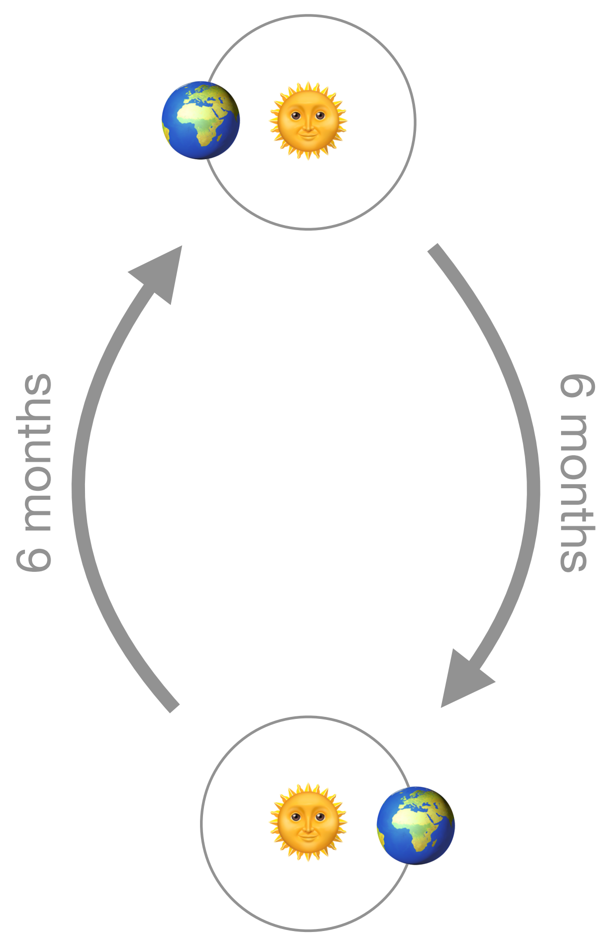

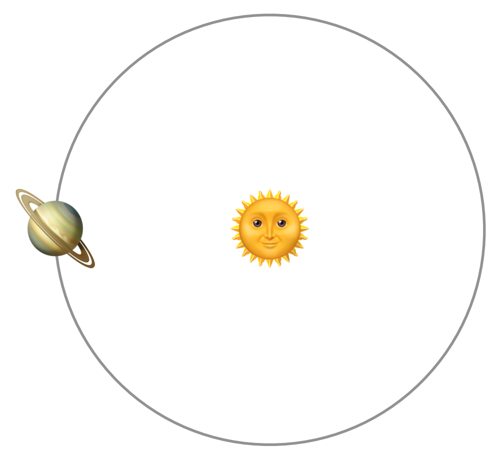

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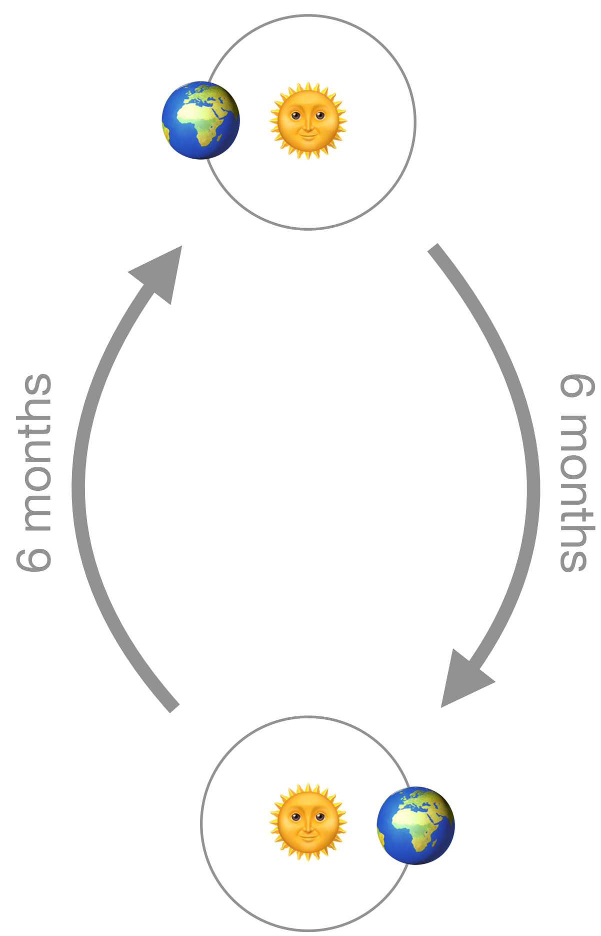

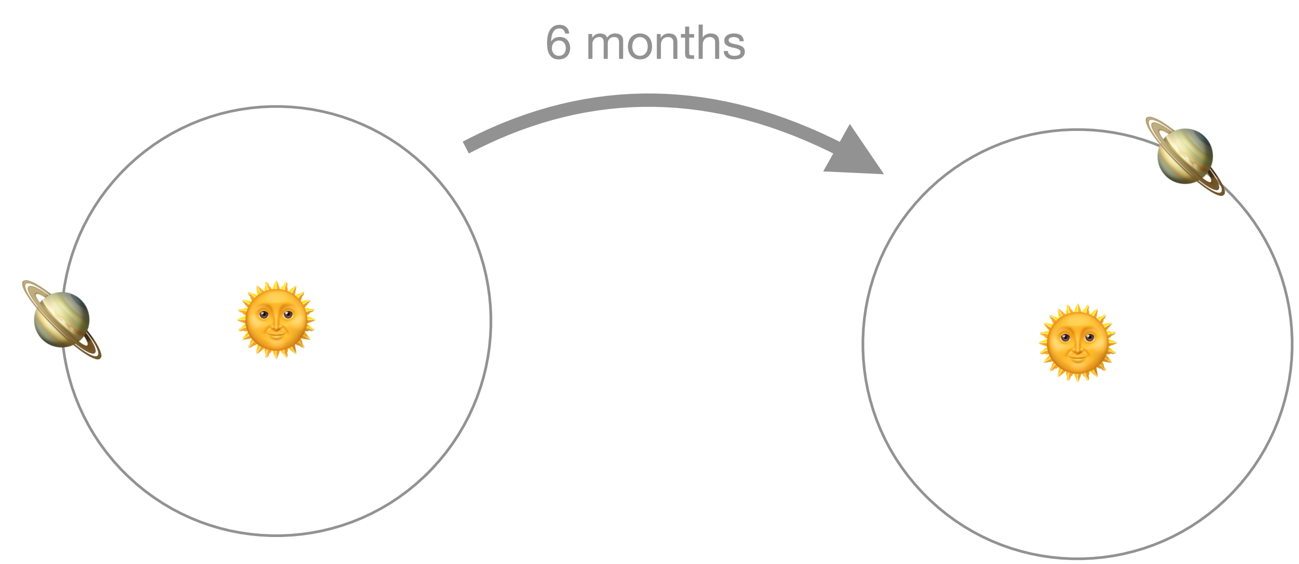

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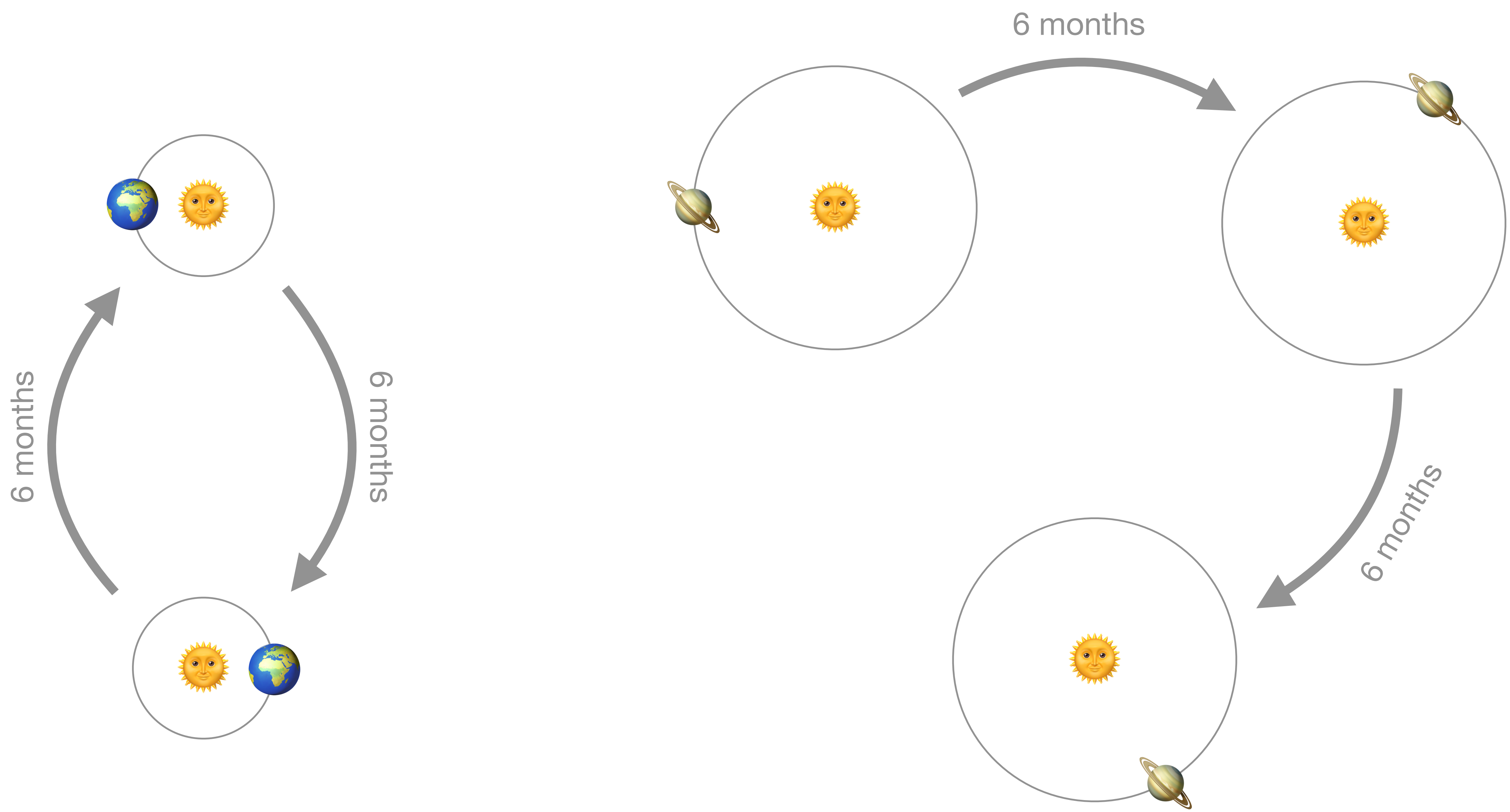

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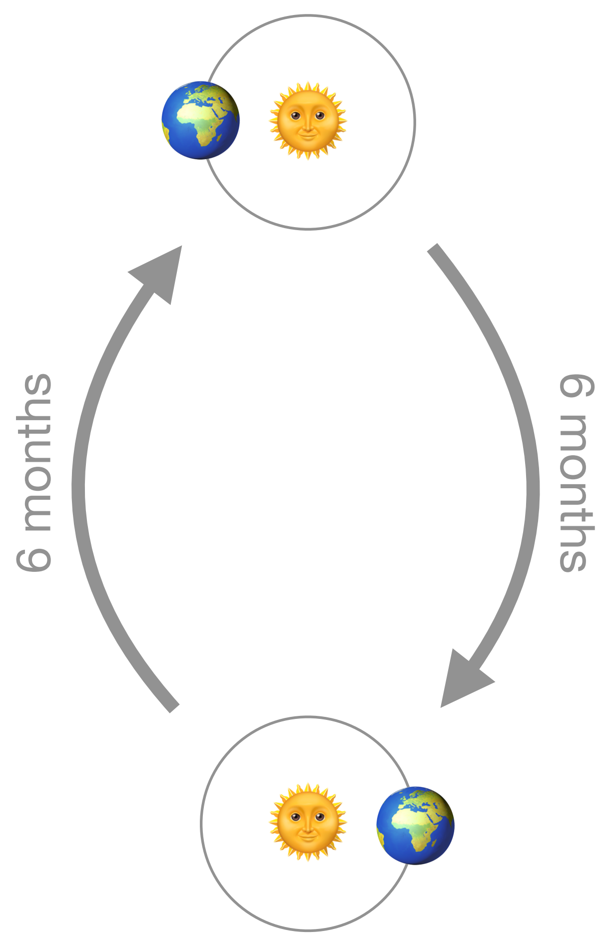

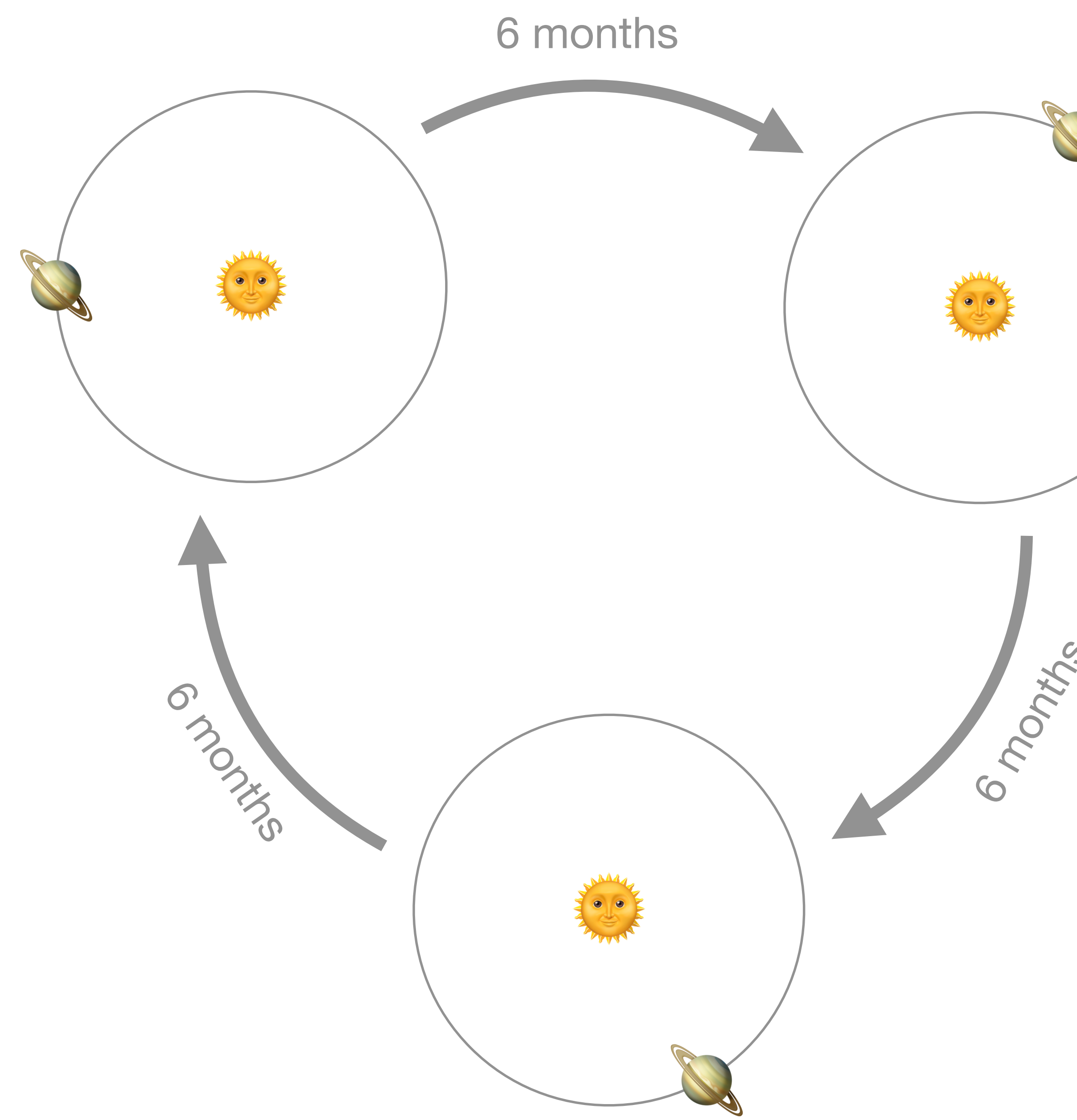

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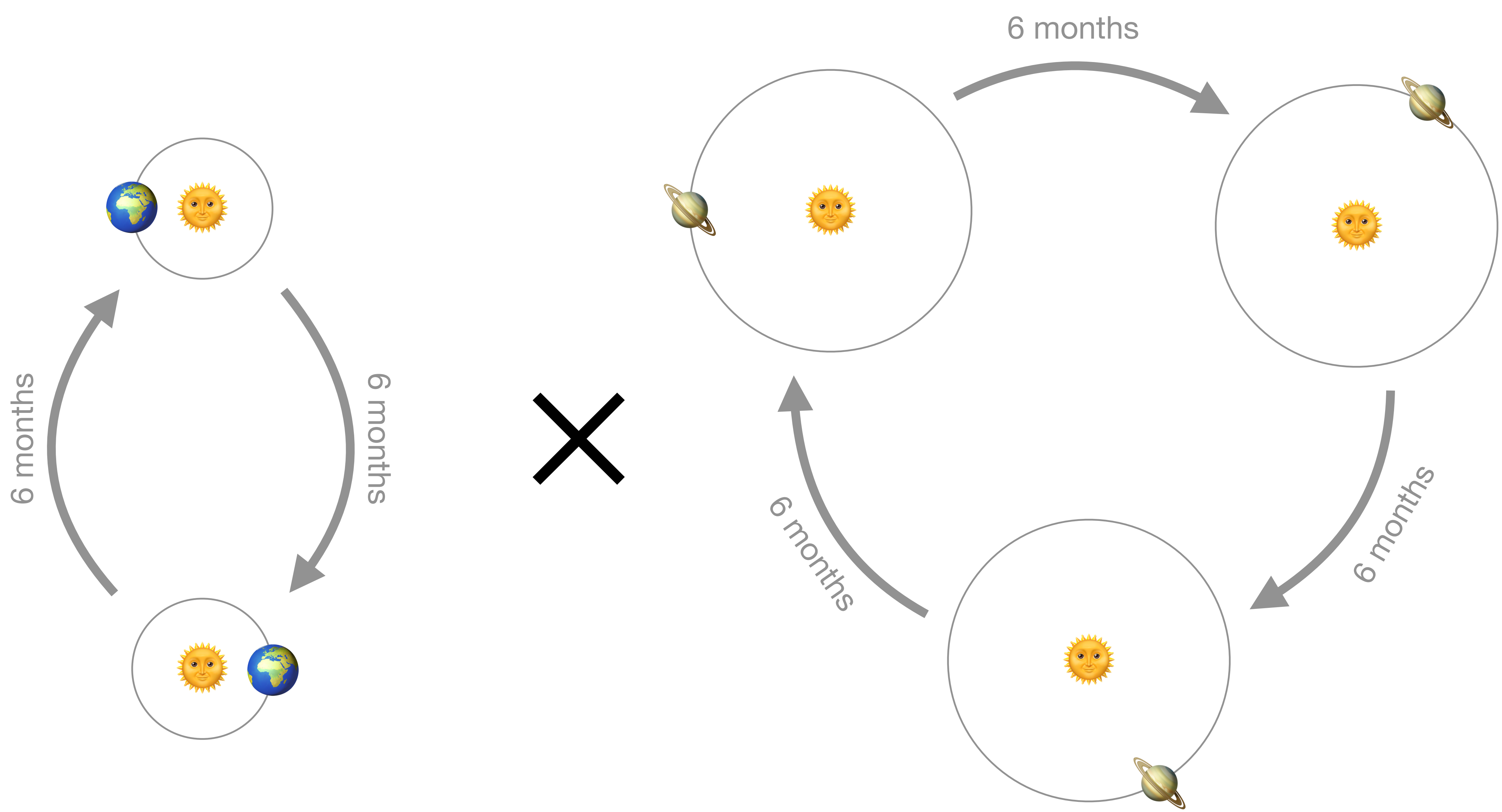
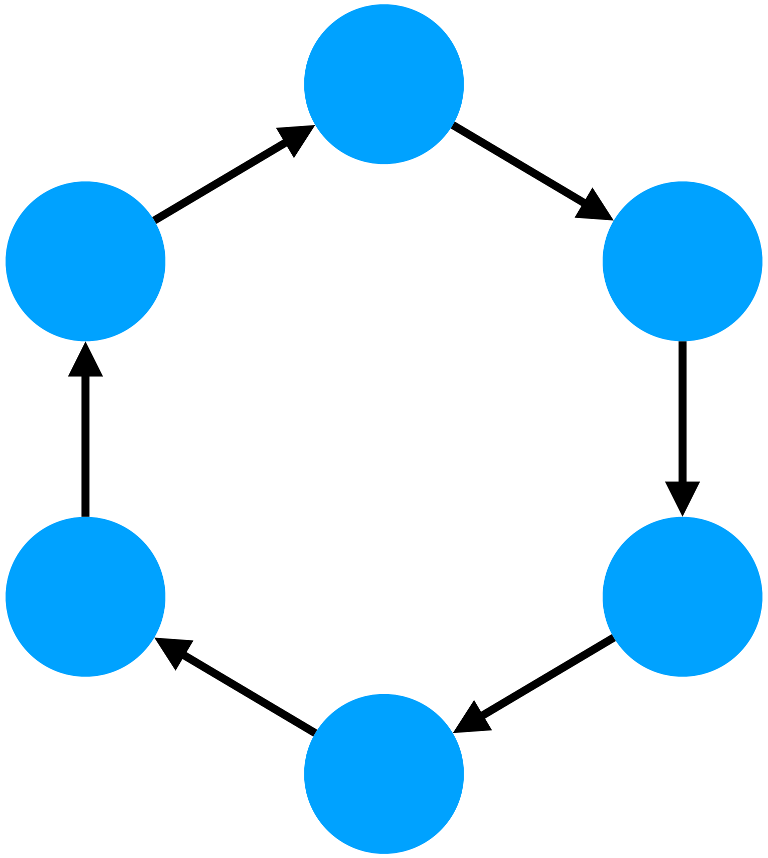


What if our instruments are less sophisticated?

Abstract evolution of the system



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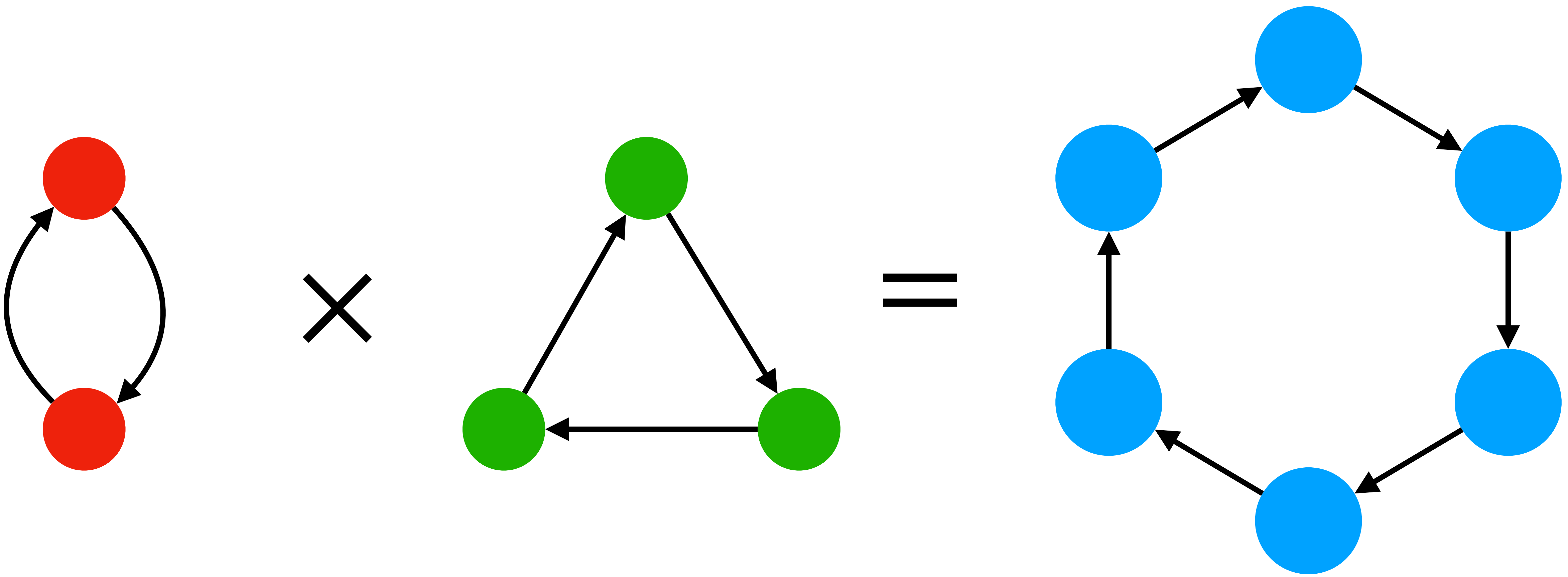

Product of dynamical systems

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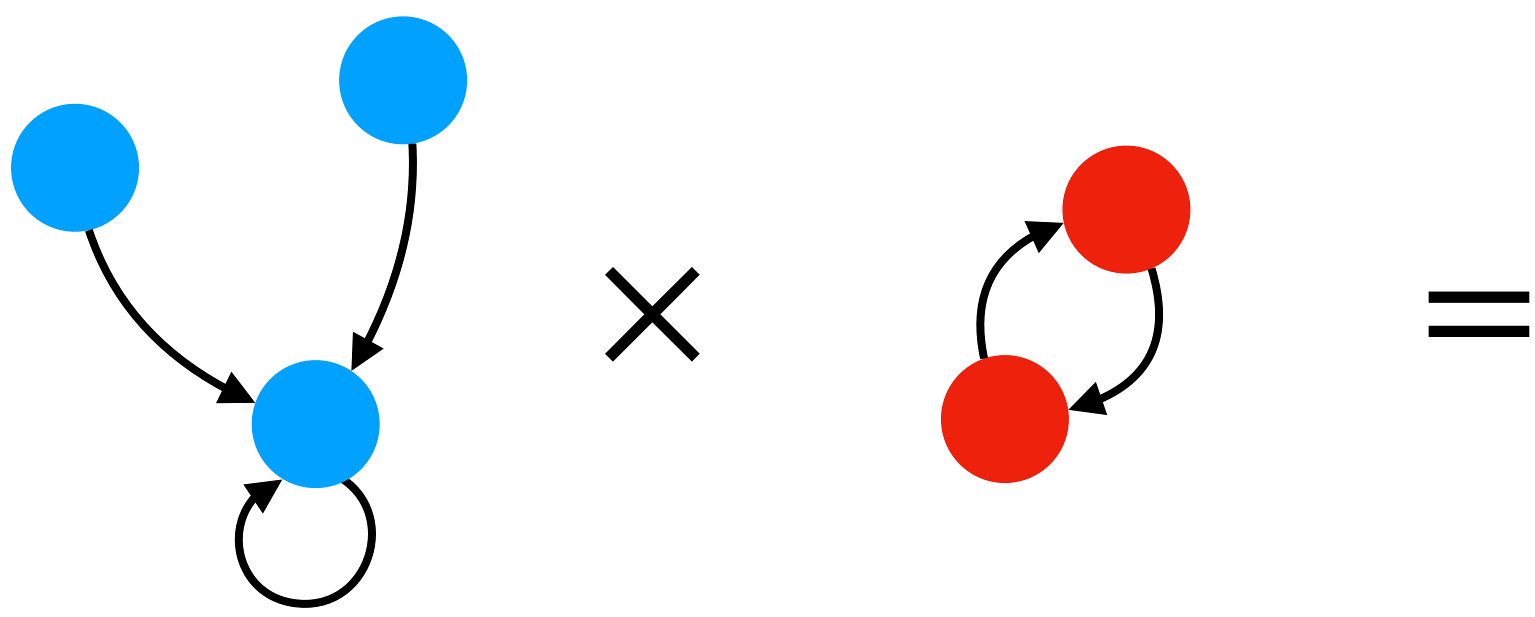
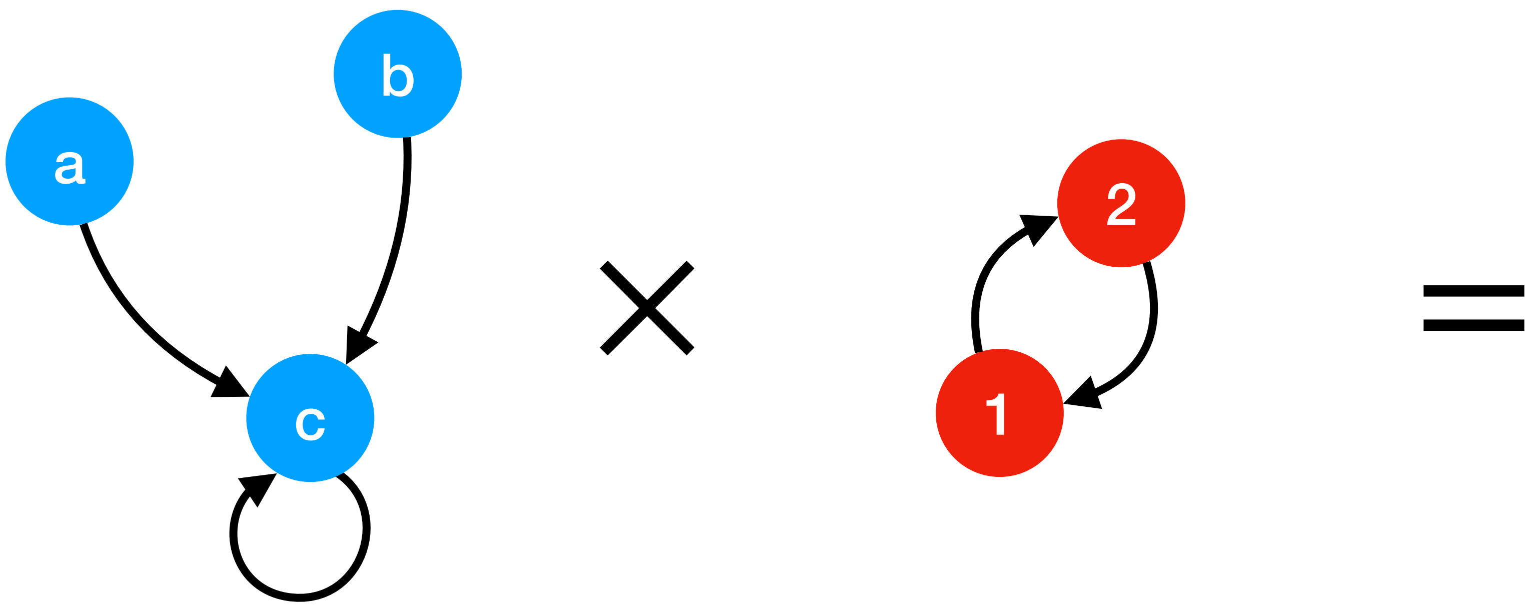
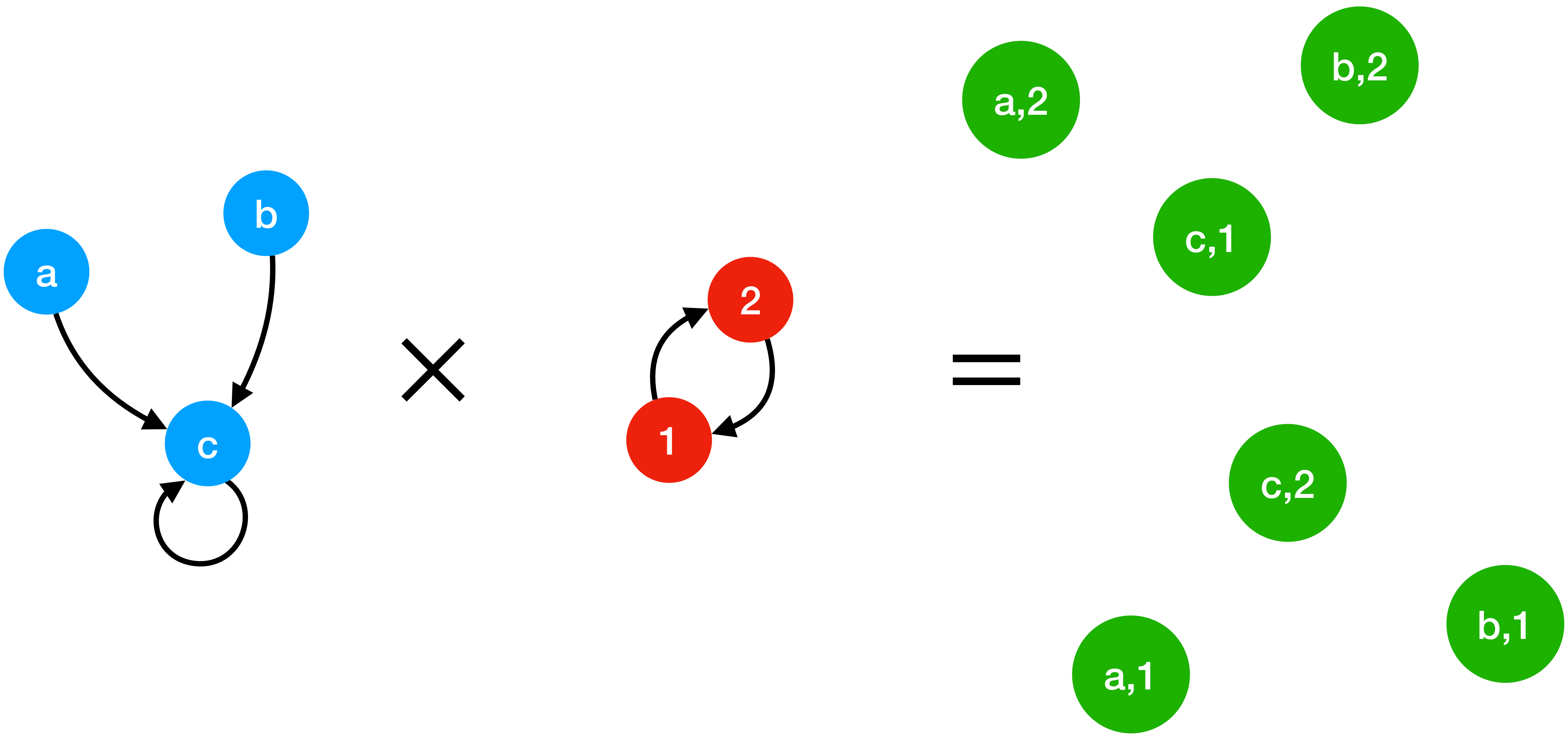


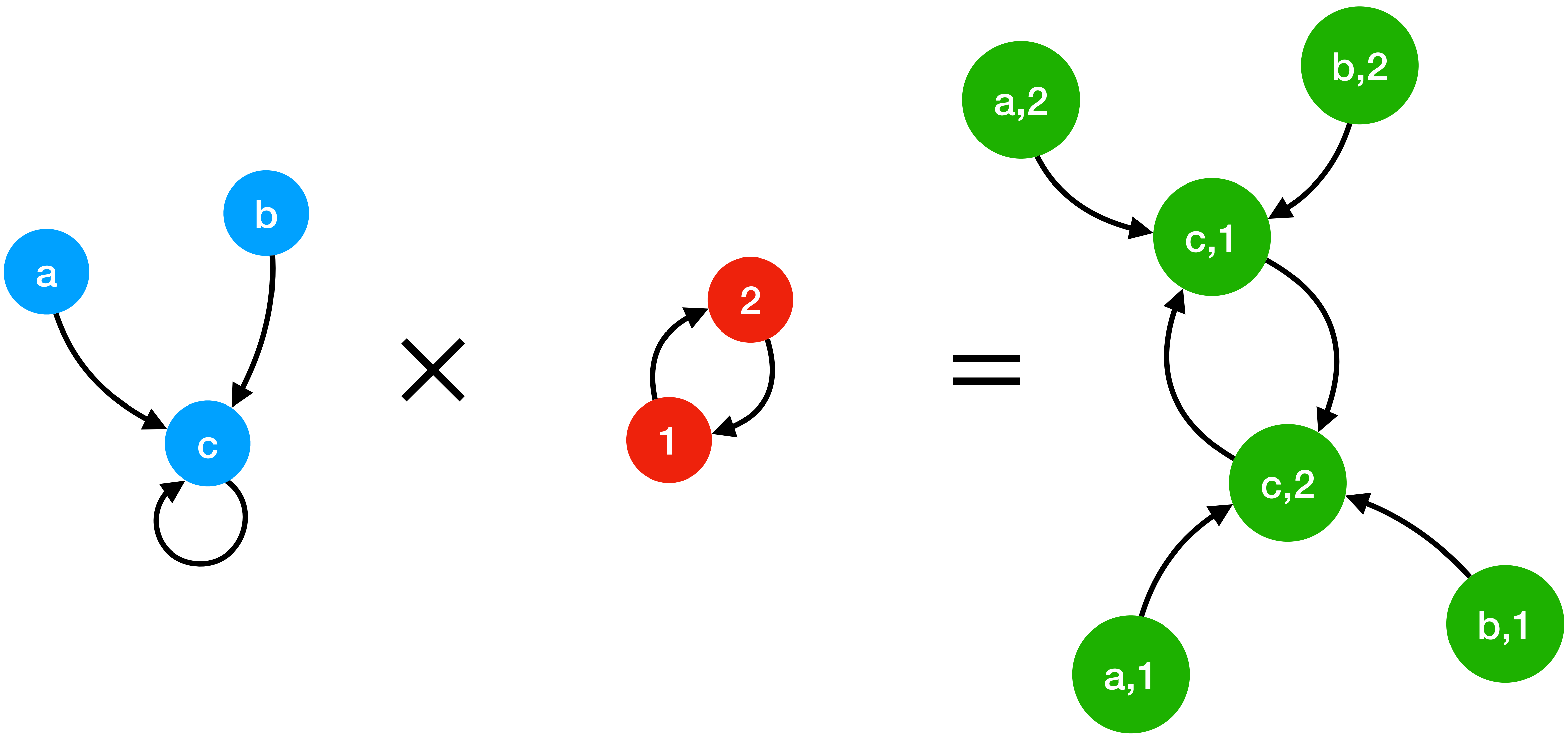
Give temporary names to the states



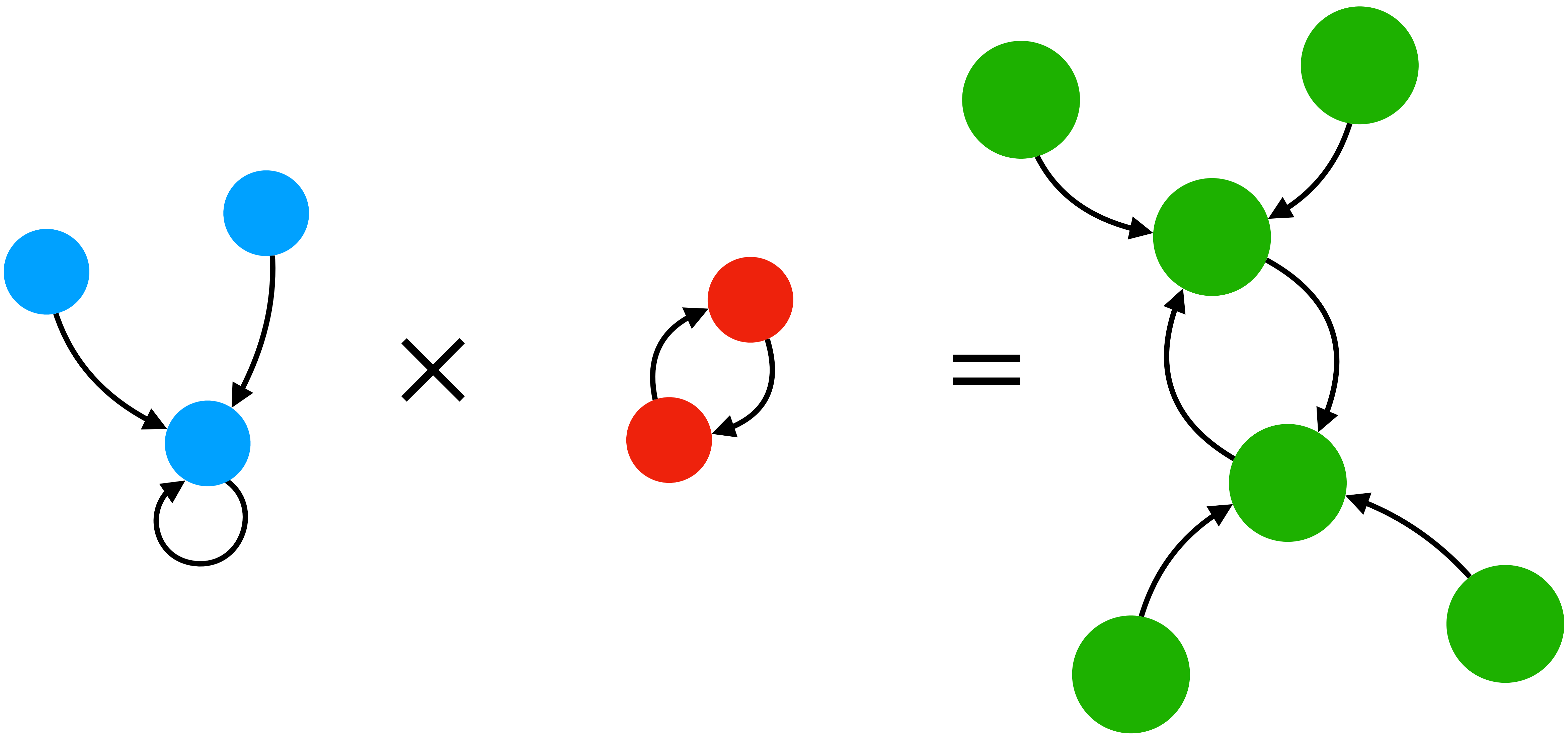
Compute the Cartesian product



Add the arcs between states

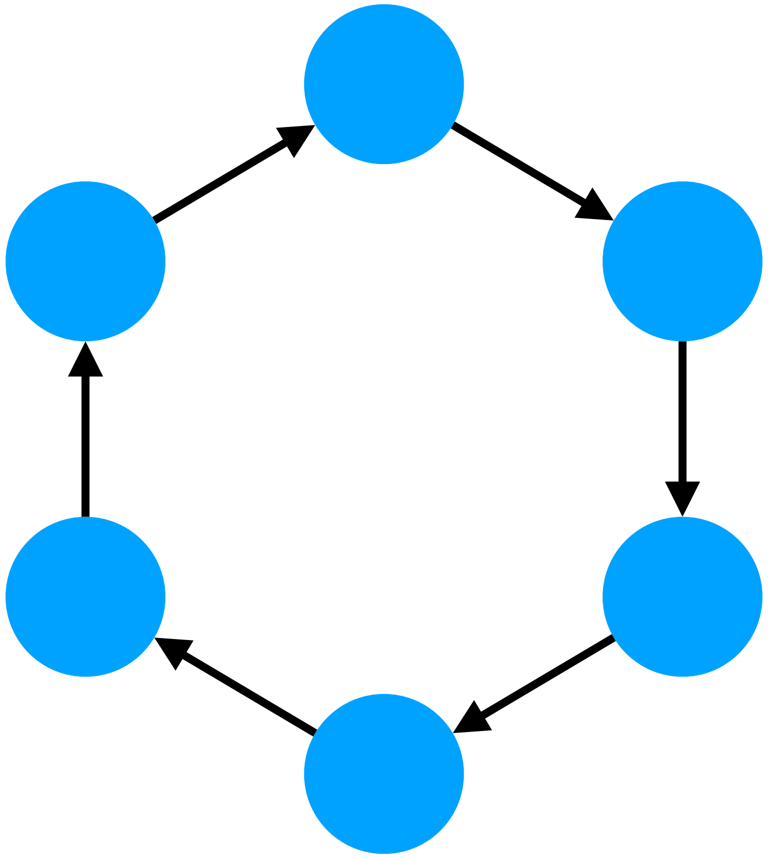


Forget the names once again

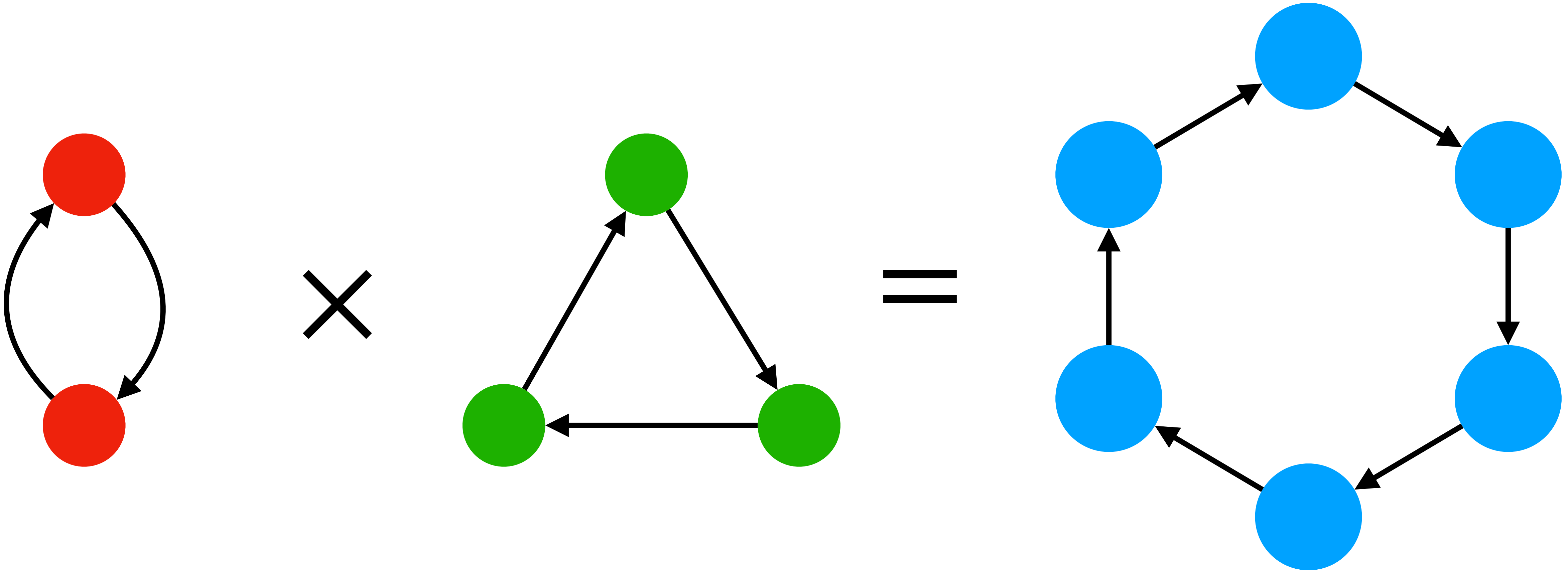


Back to our planetary system

Decomposition



Decomposition



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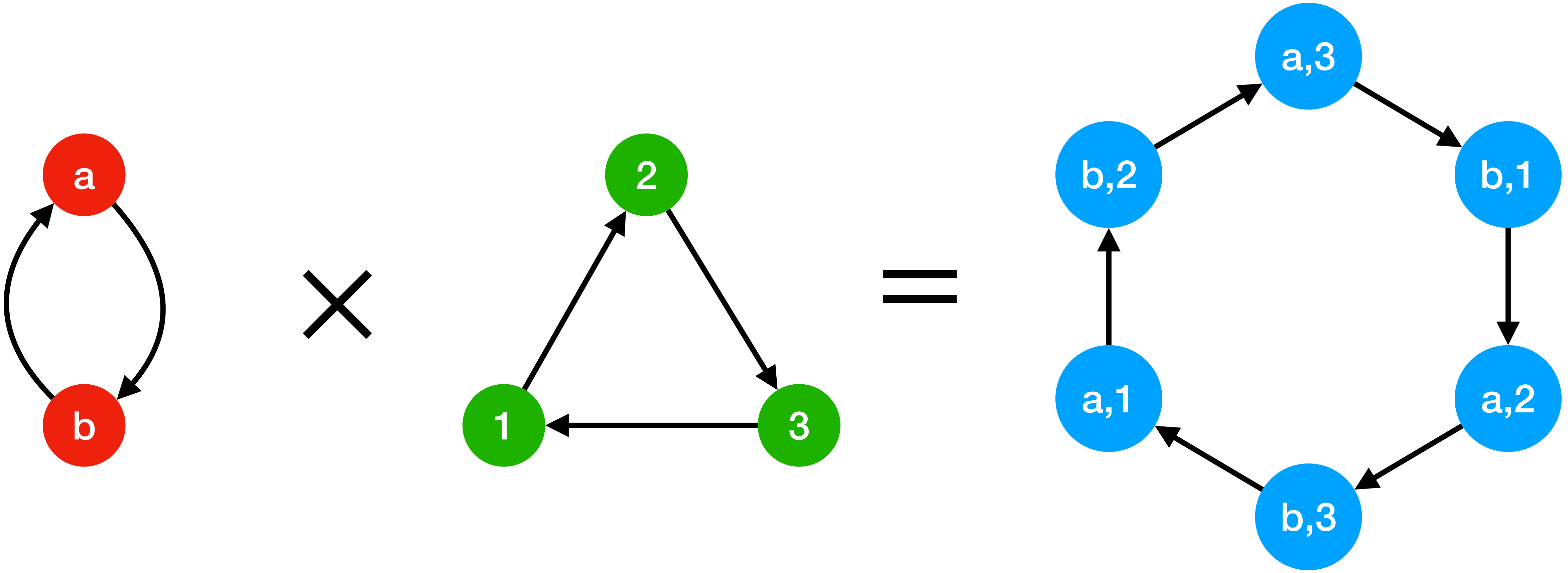
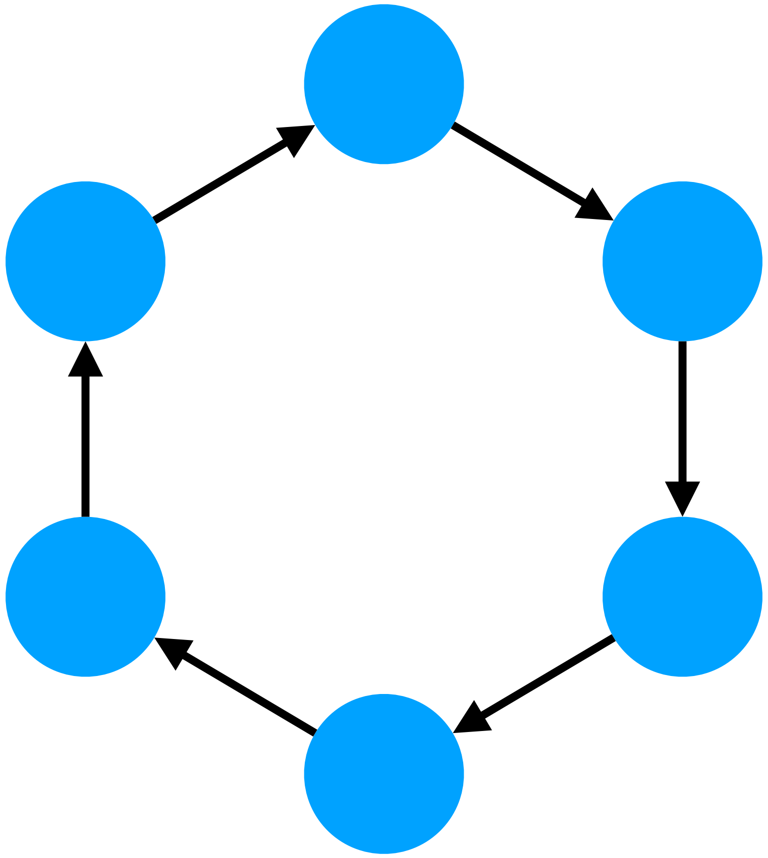
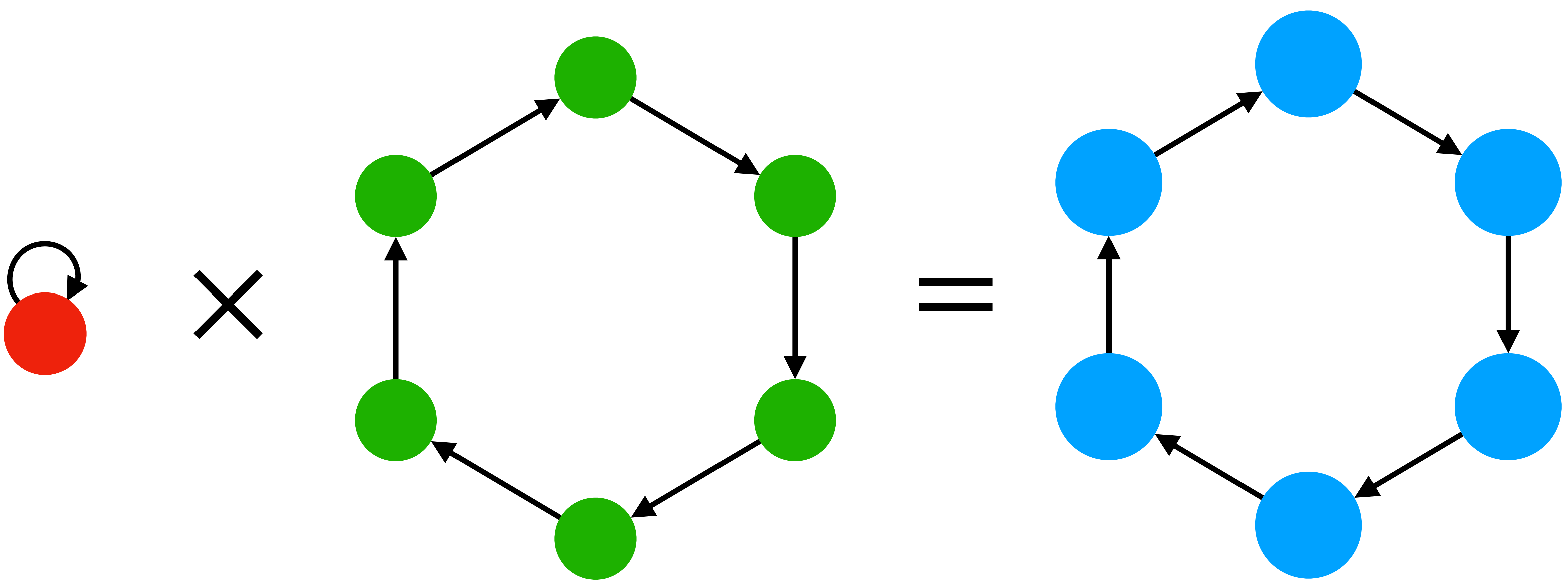


Any other decomposition?

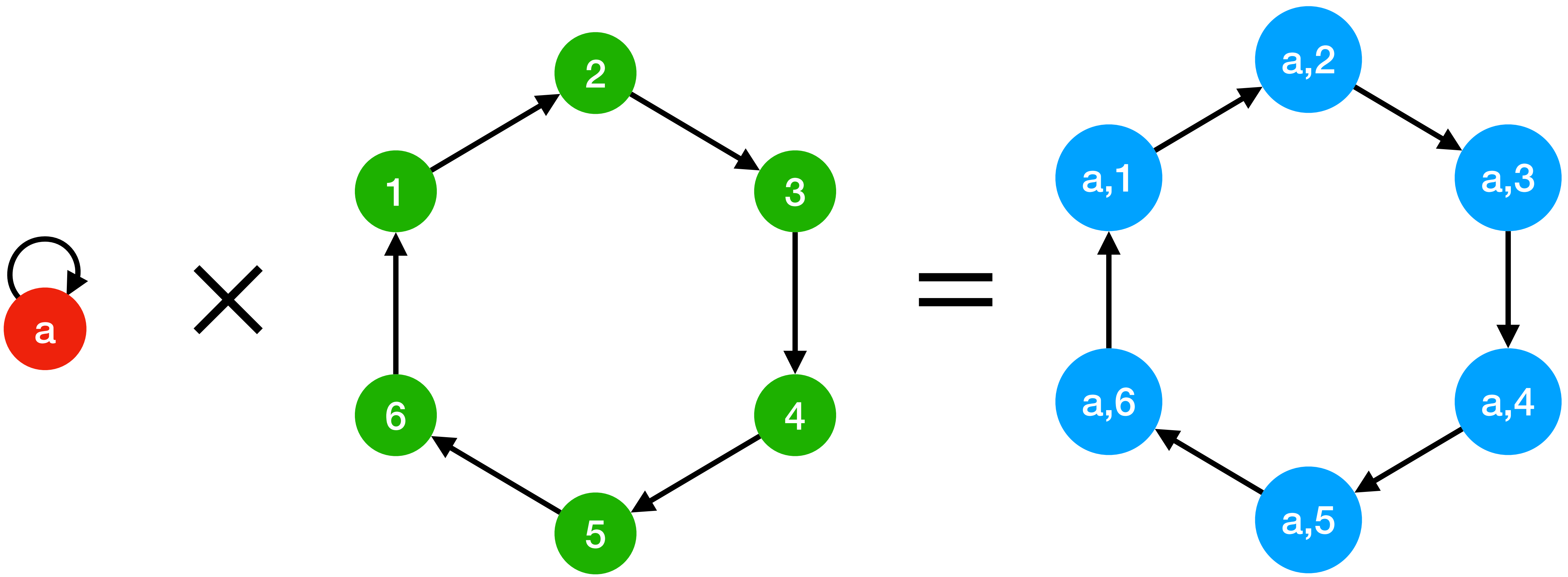
Another decomposition

$=$ 

Another decomposition



Another decomposition



More concretely…

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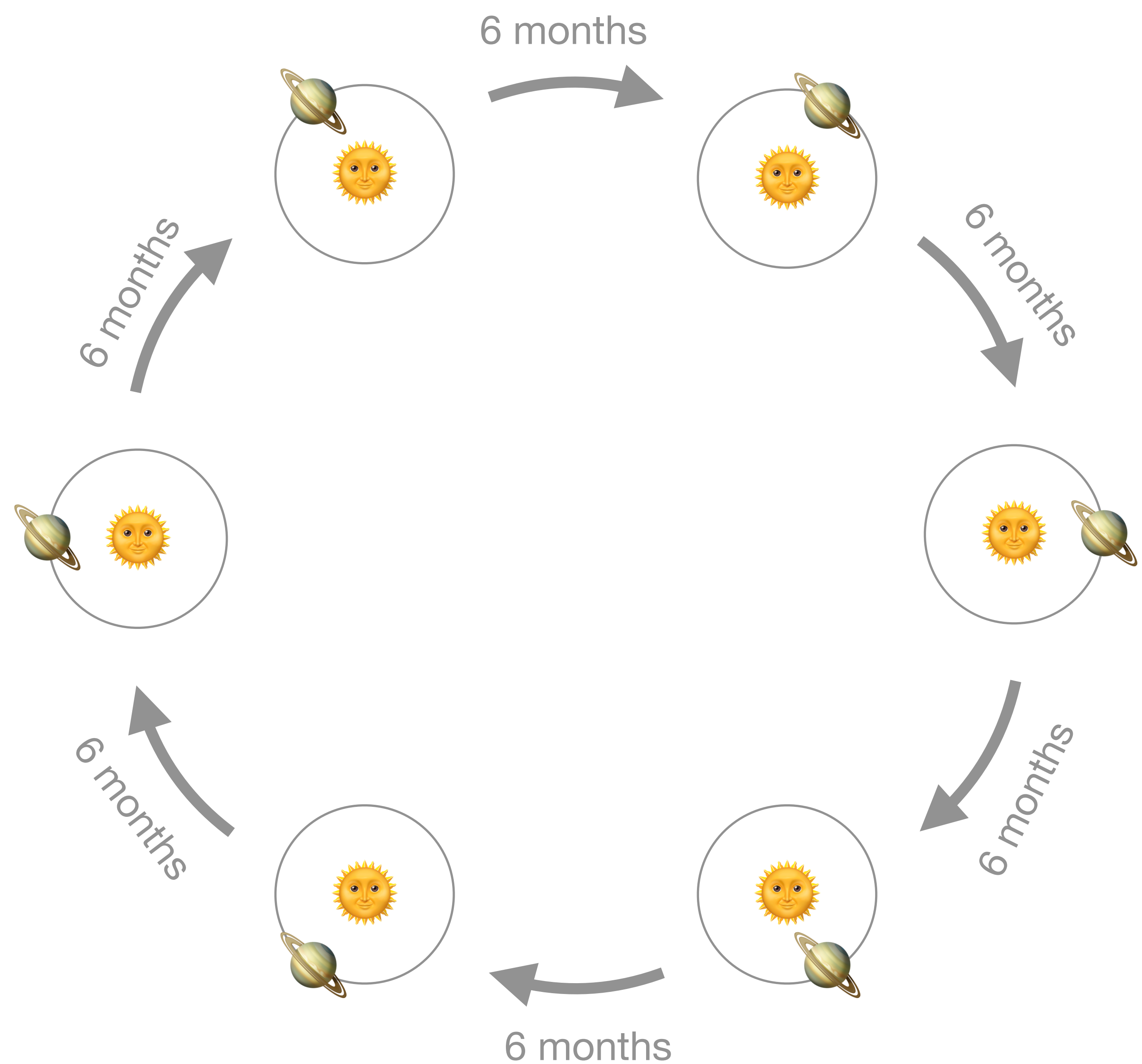

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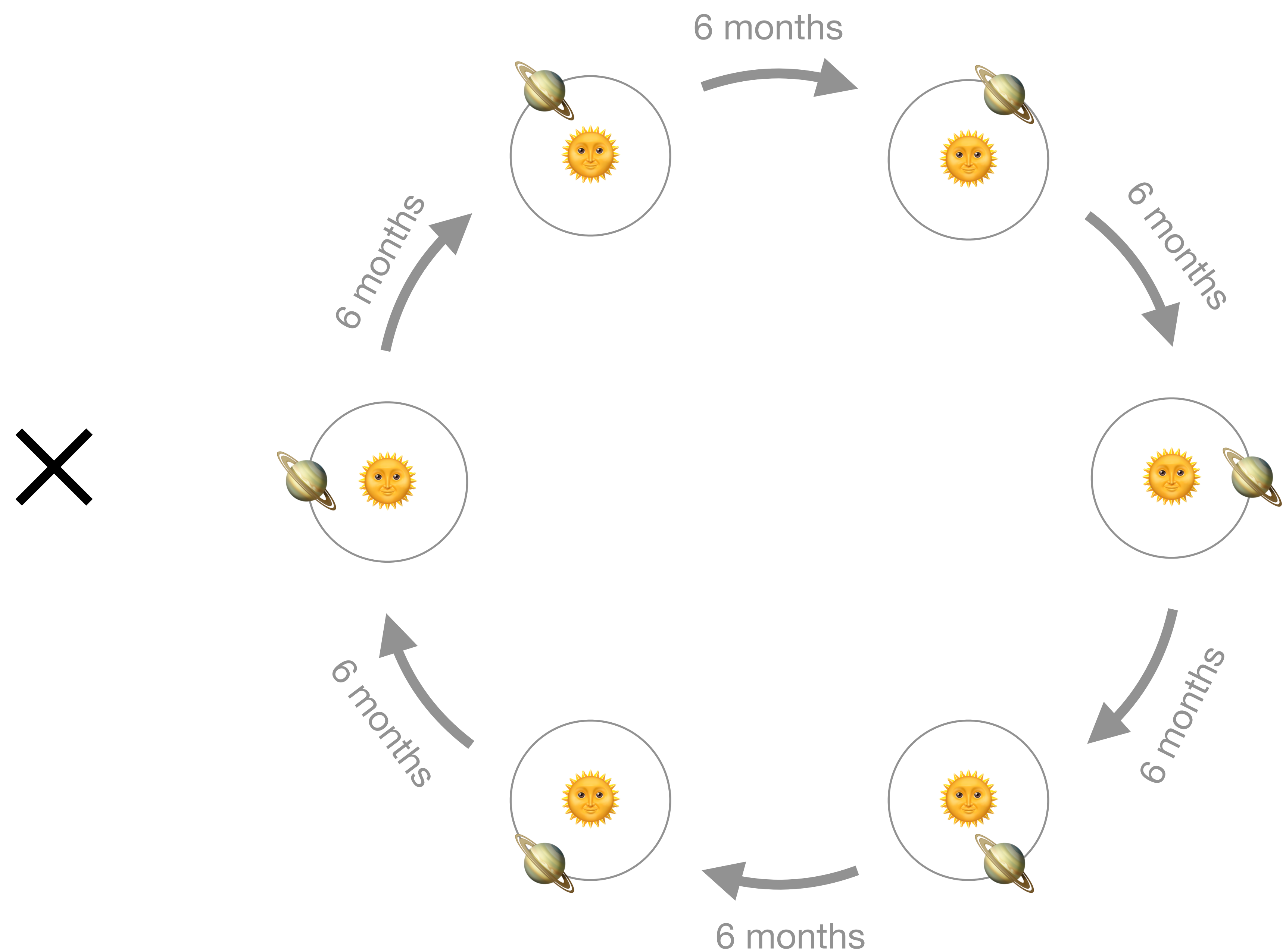

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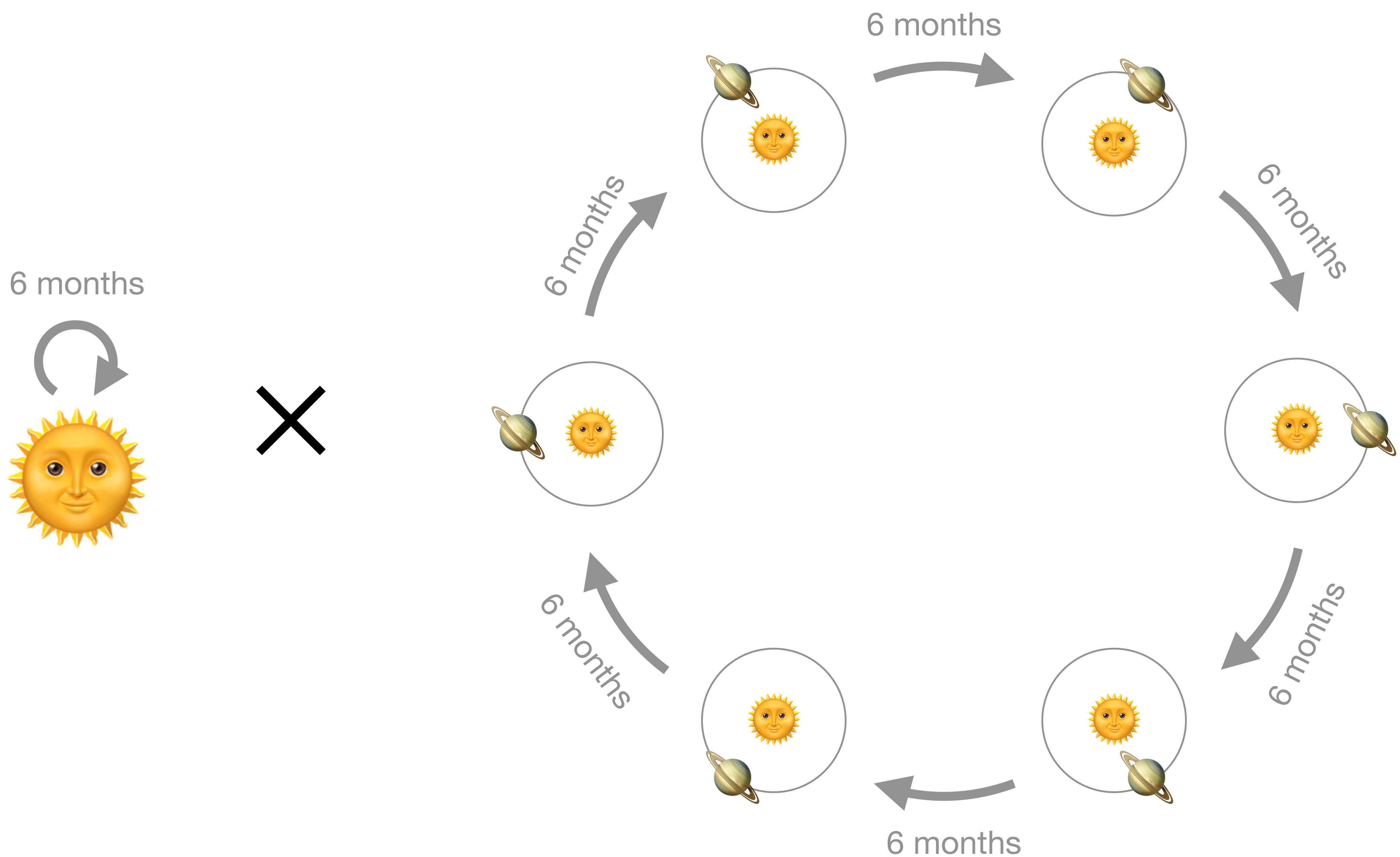
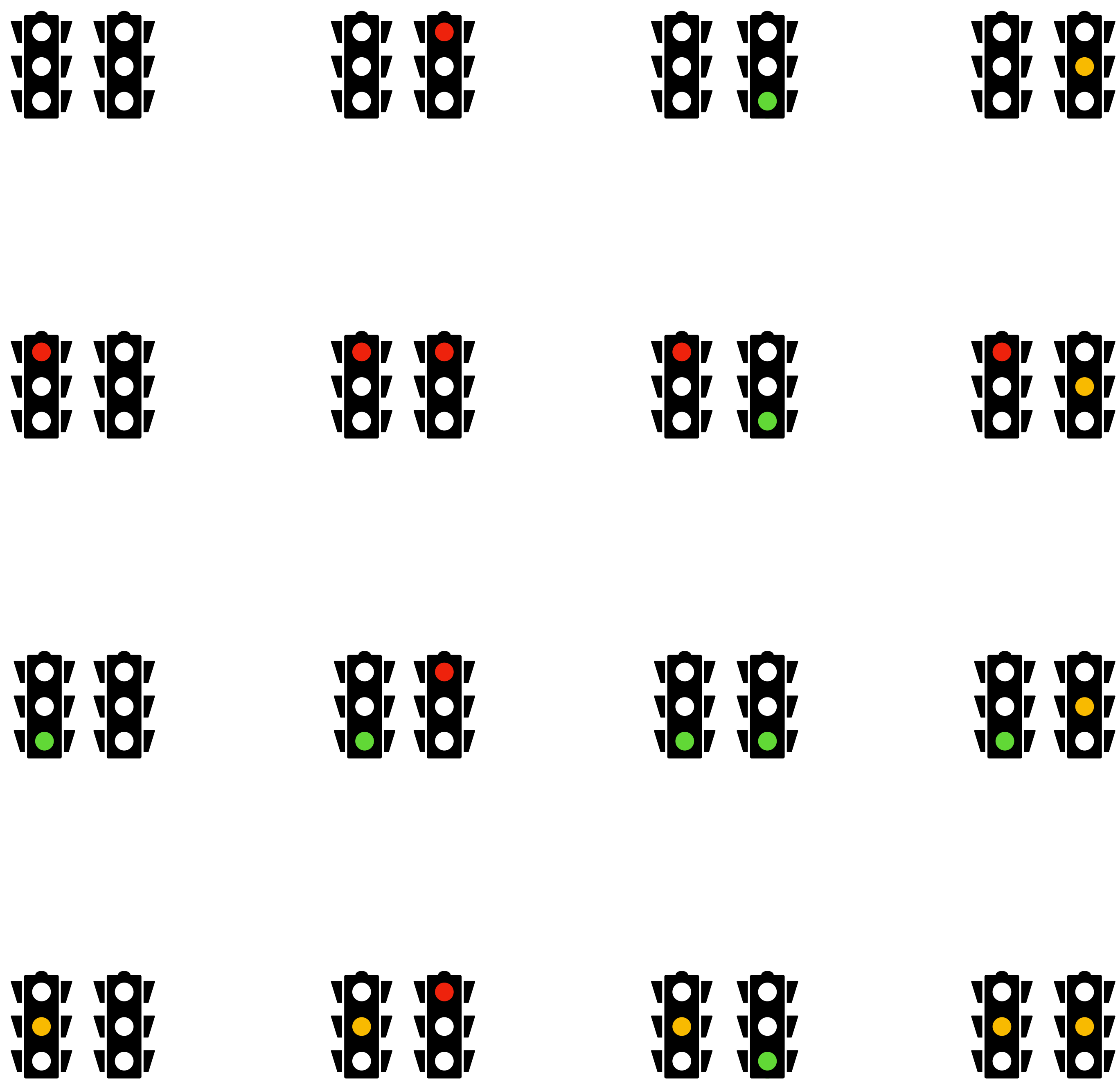


Untangling complex systems

Traffic lights at a crossroads



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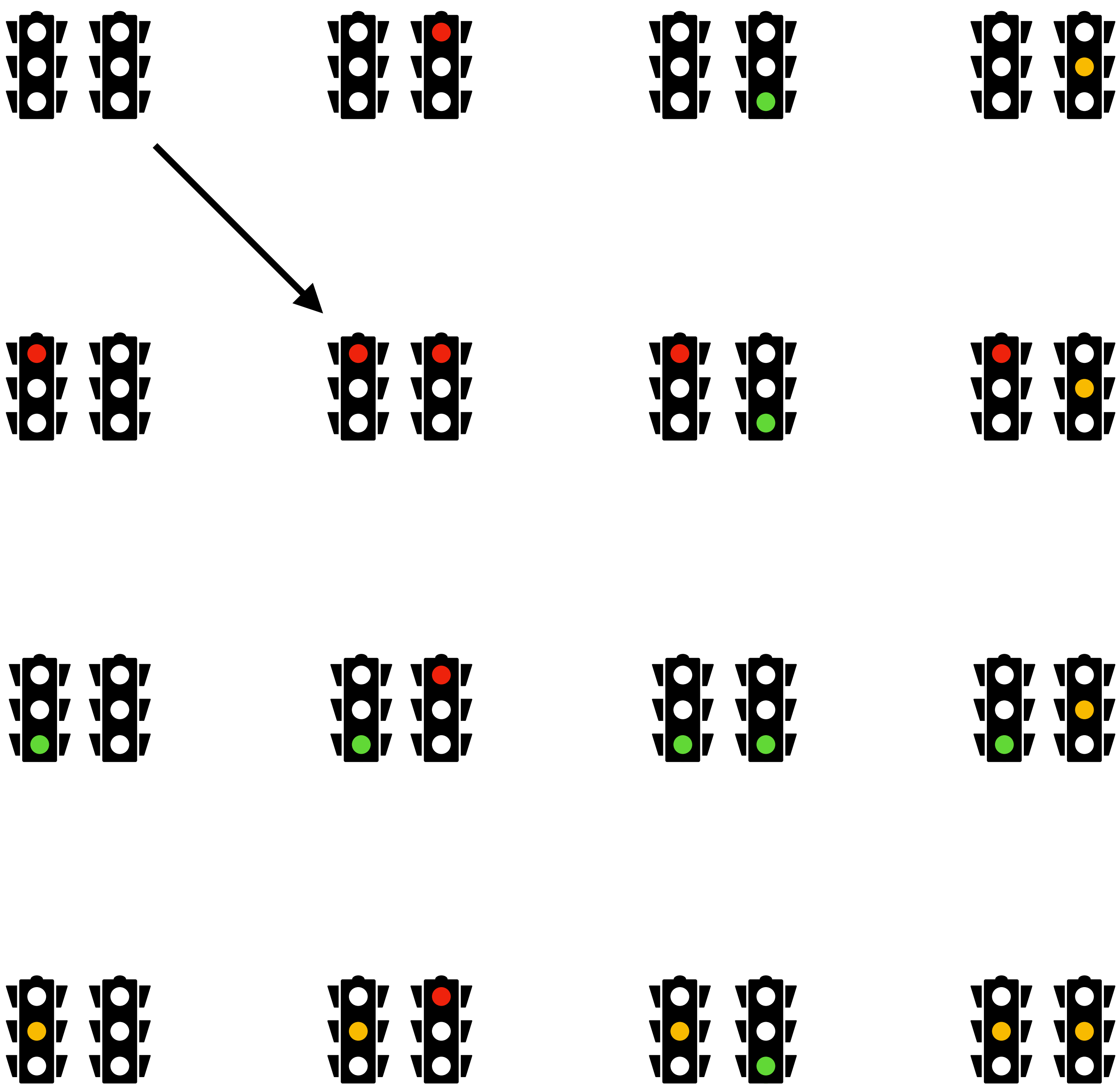

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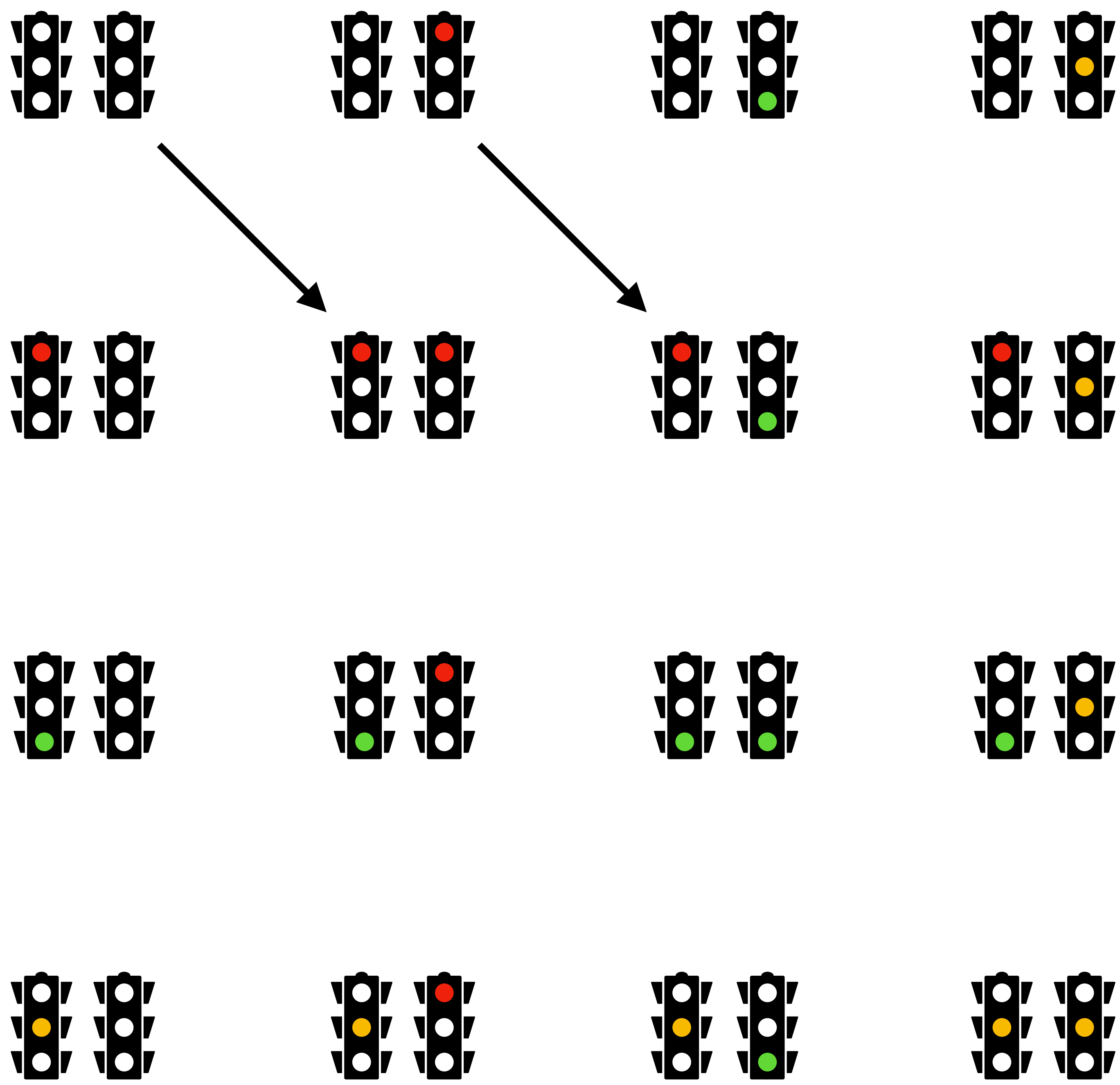

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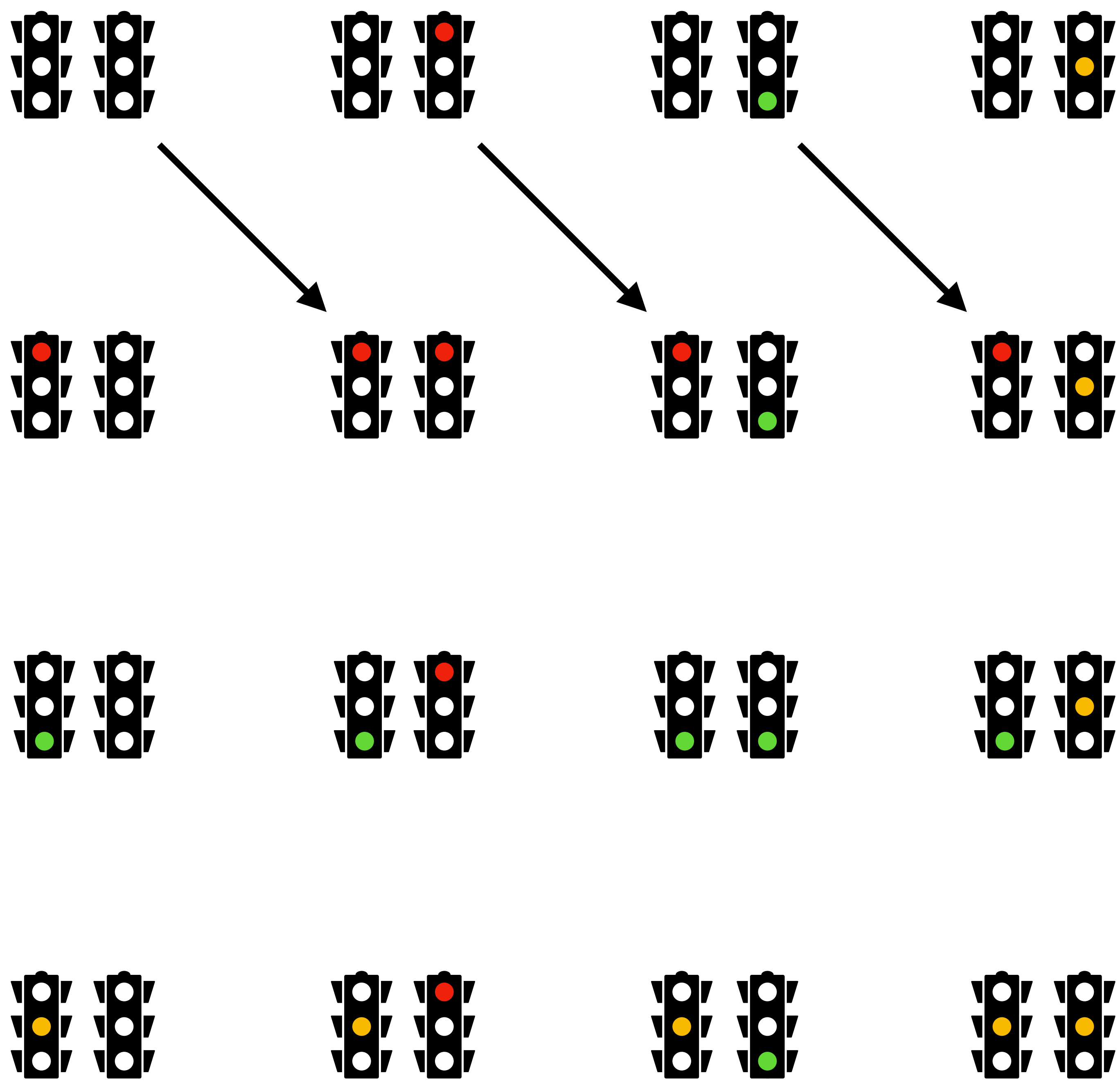

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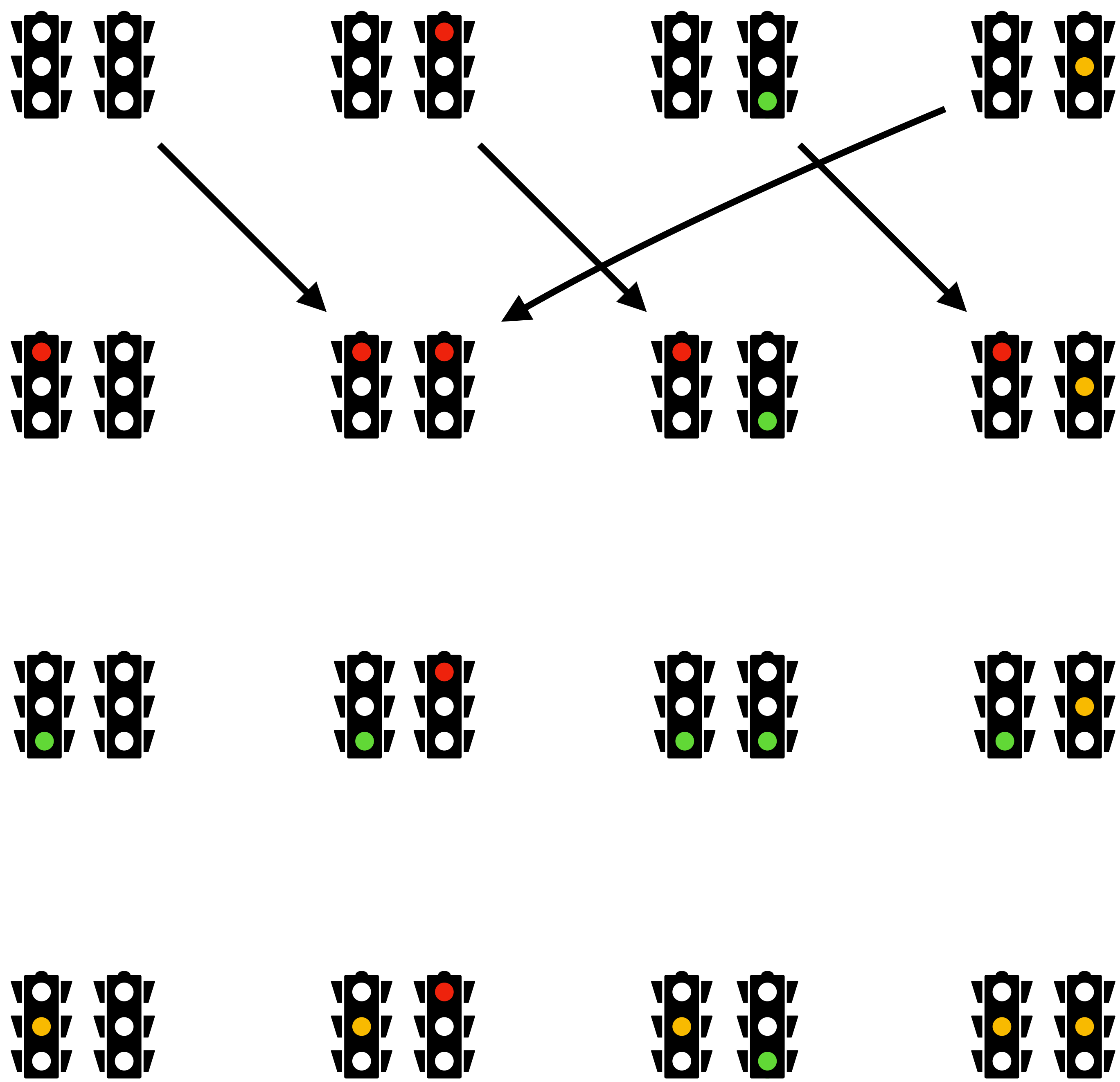

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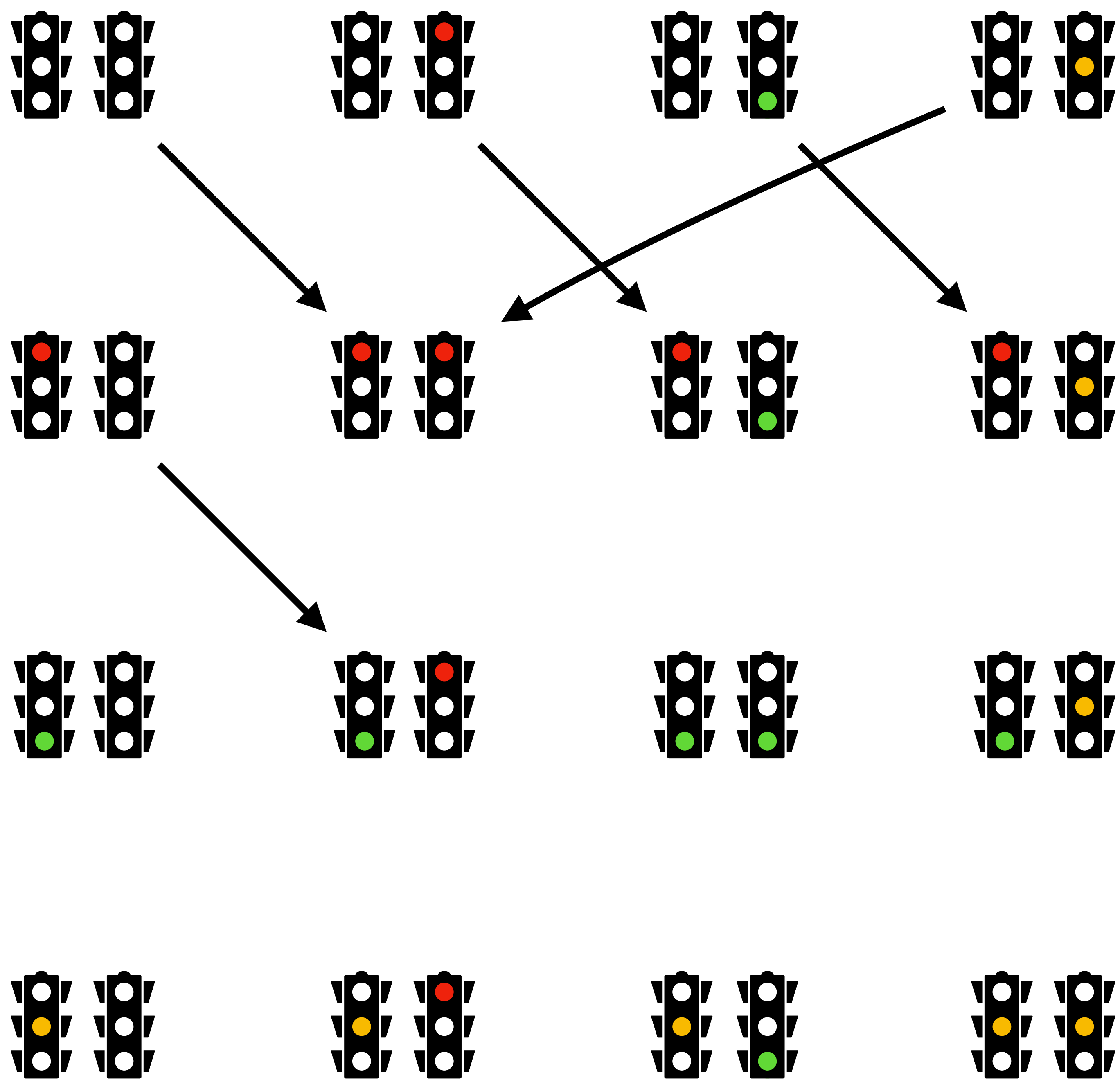

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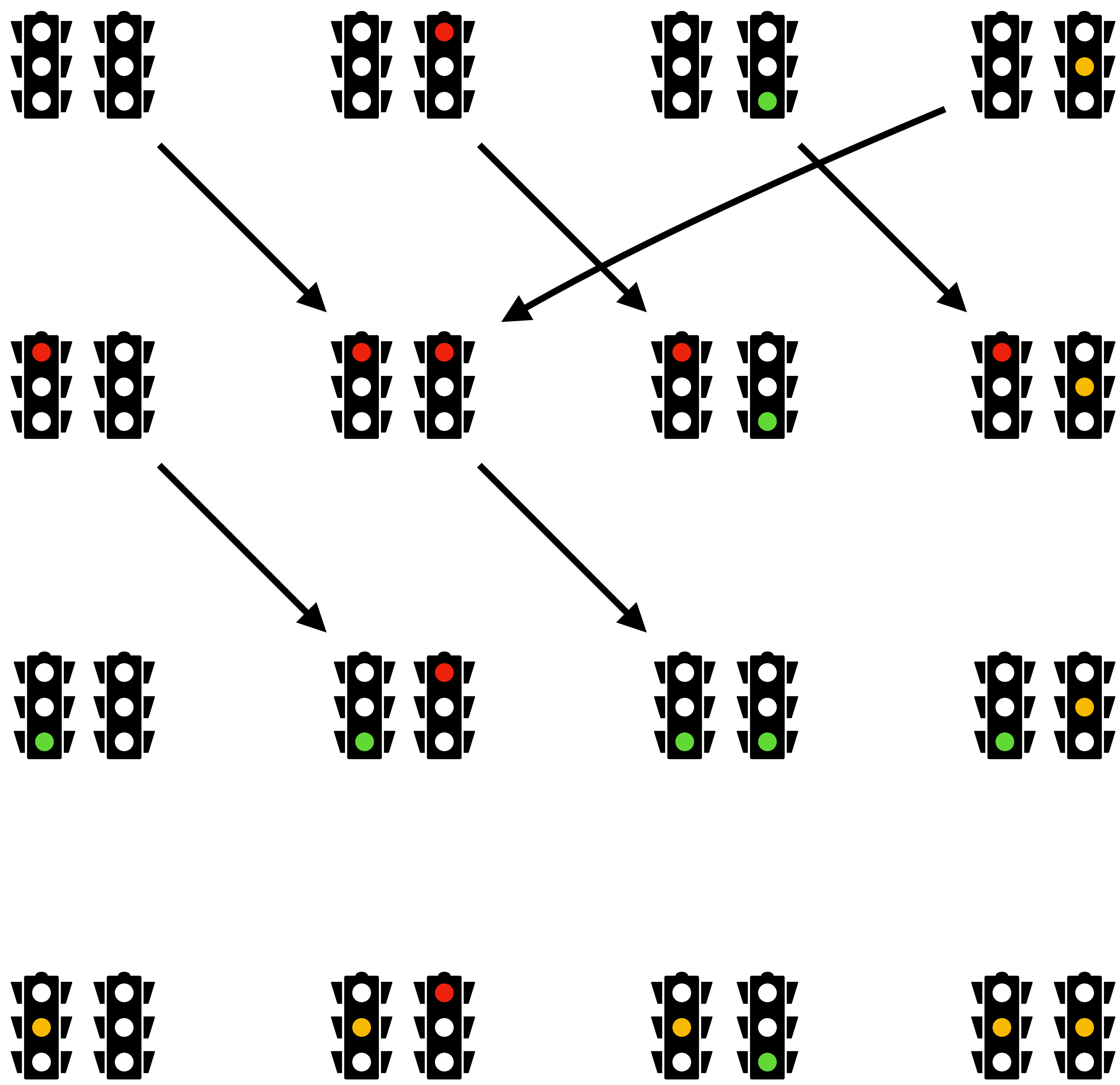

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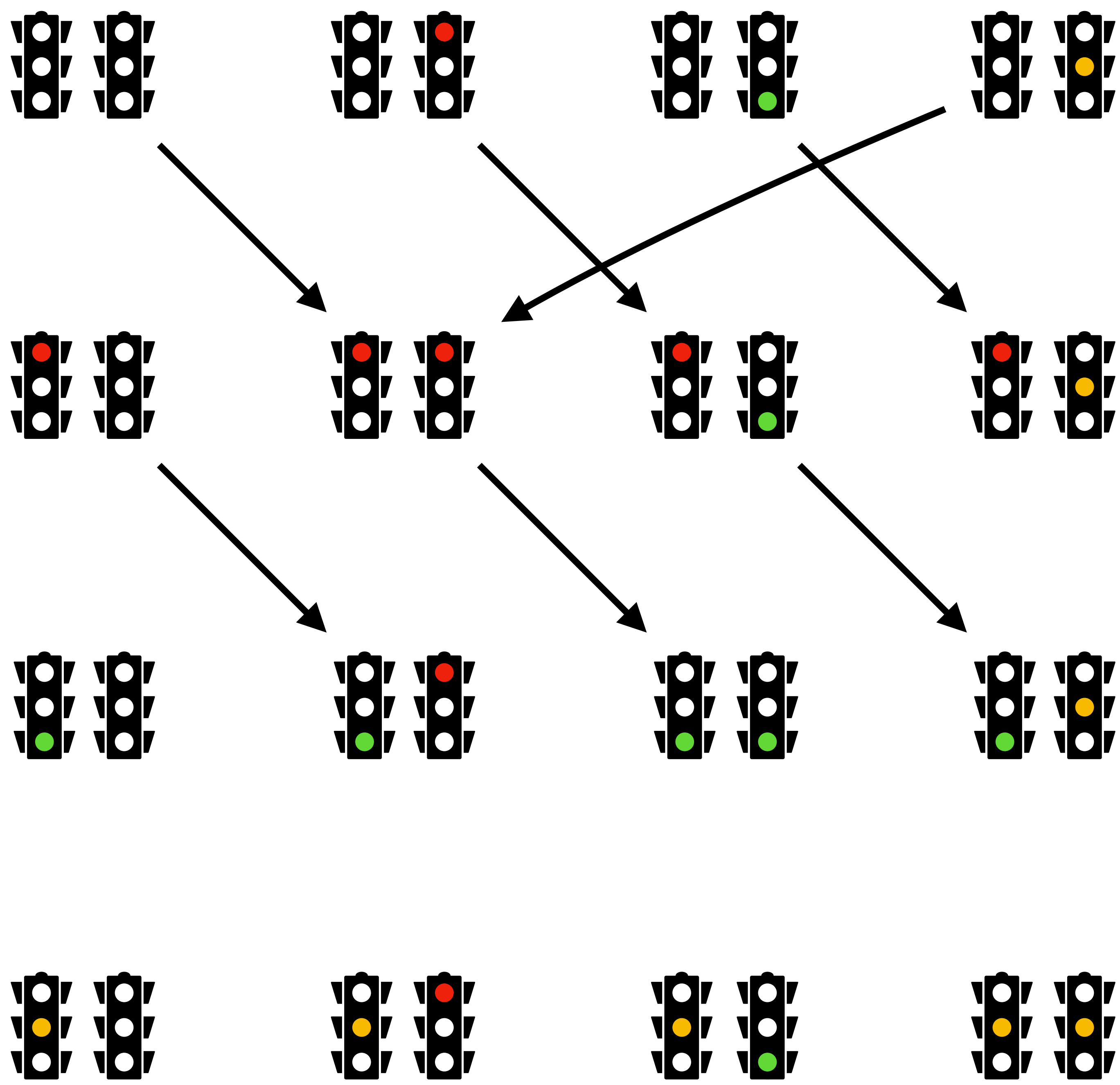

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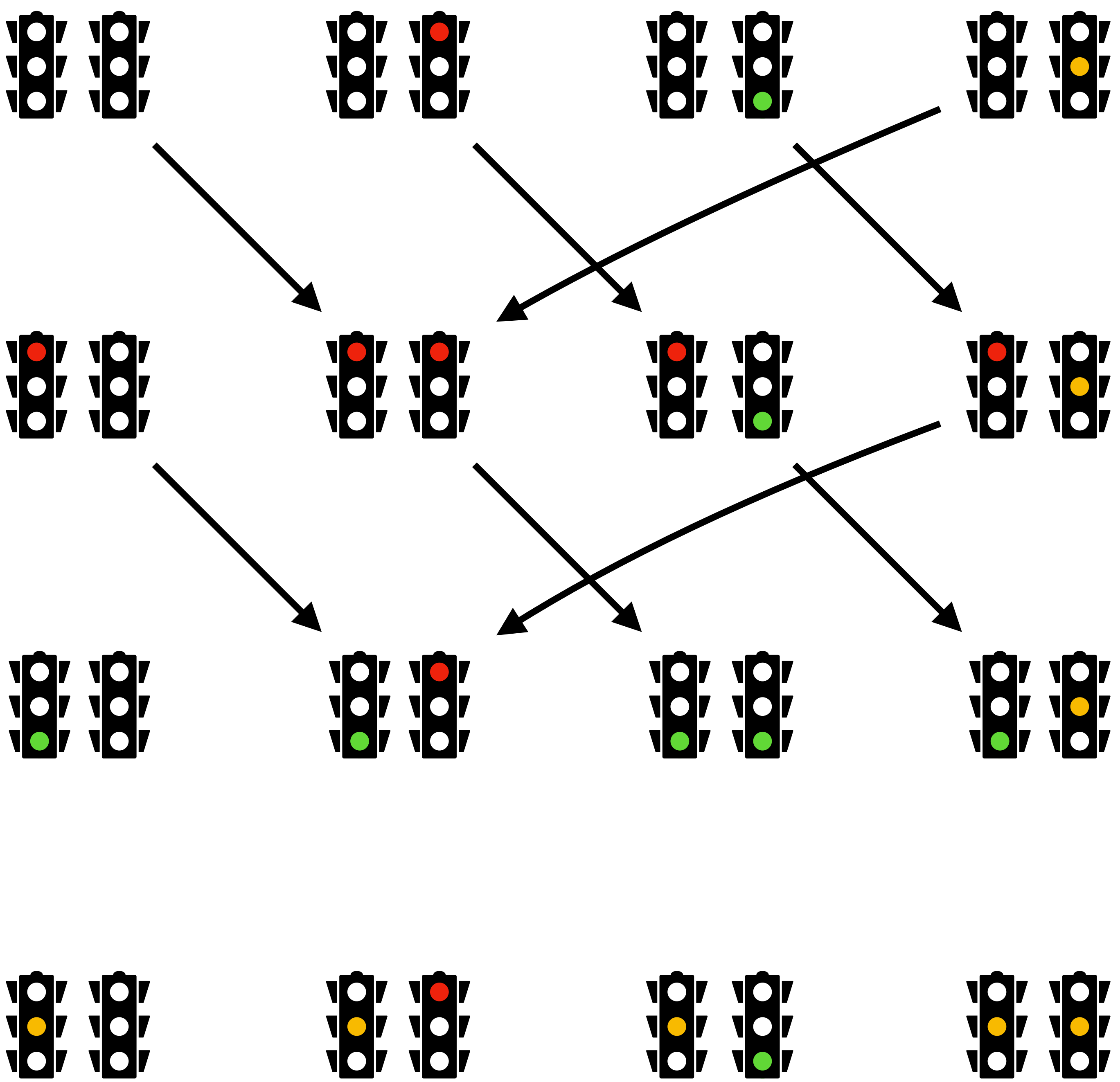

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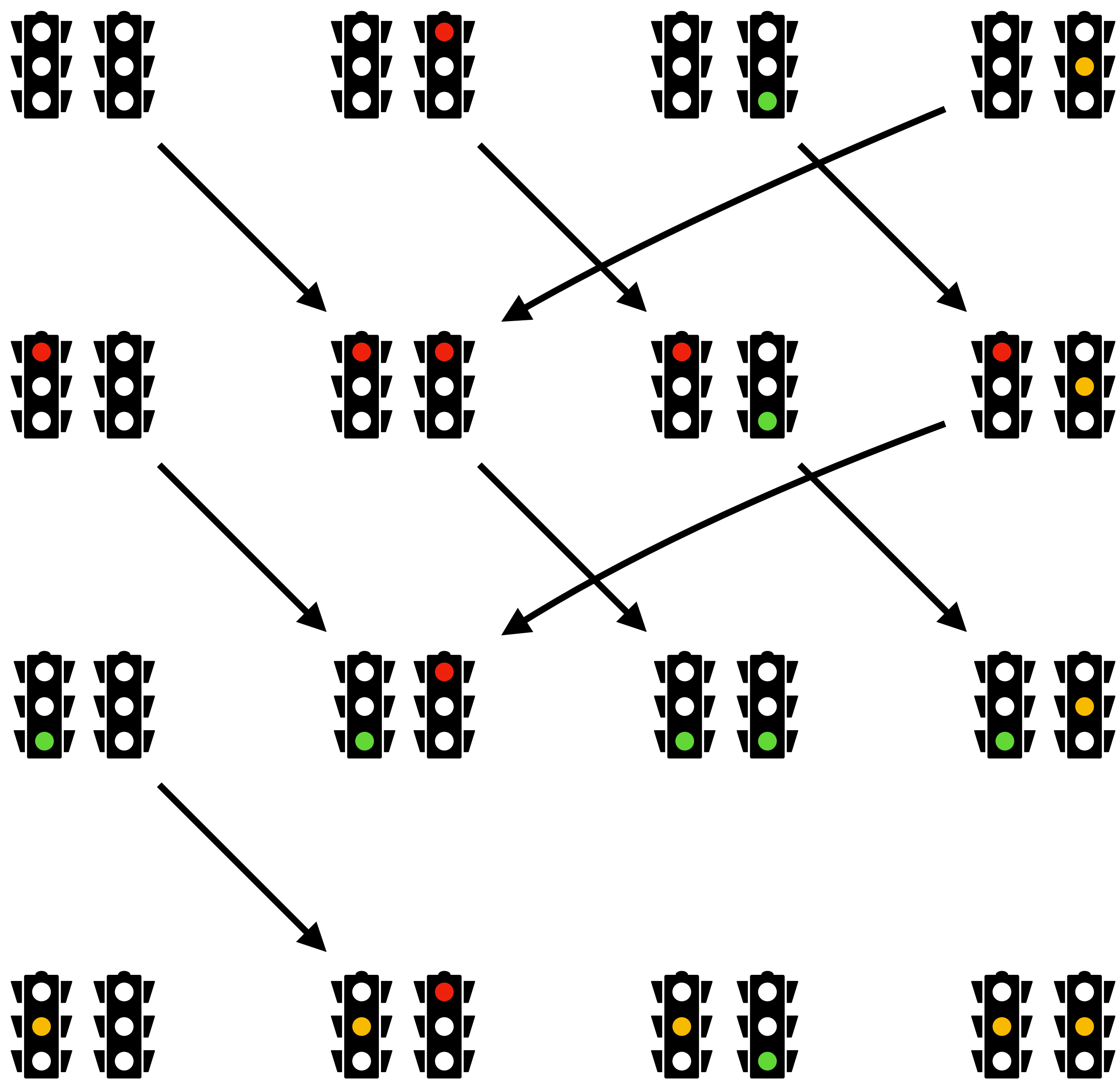

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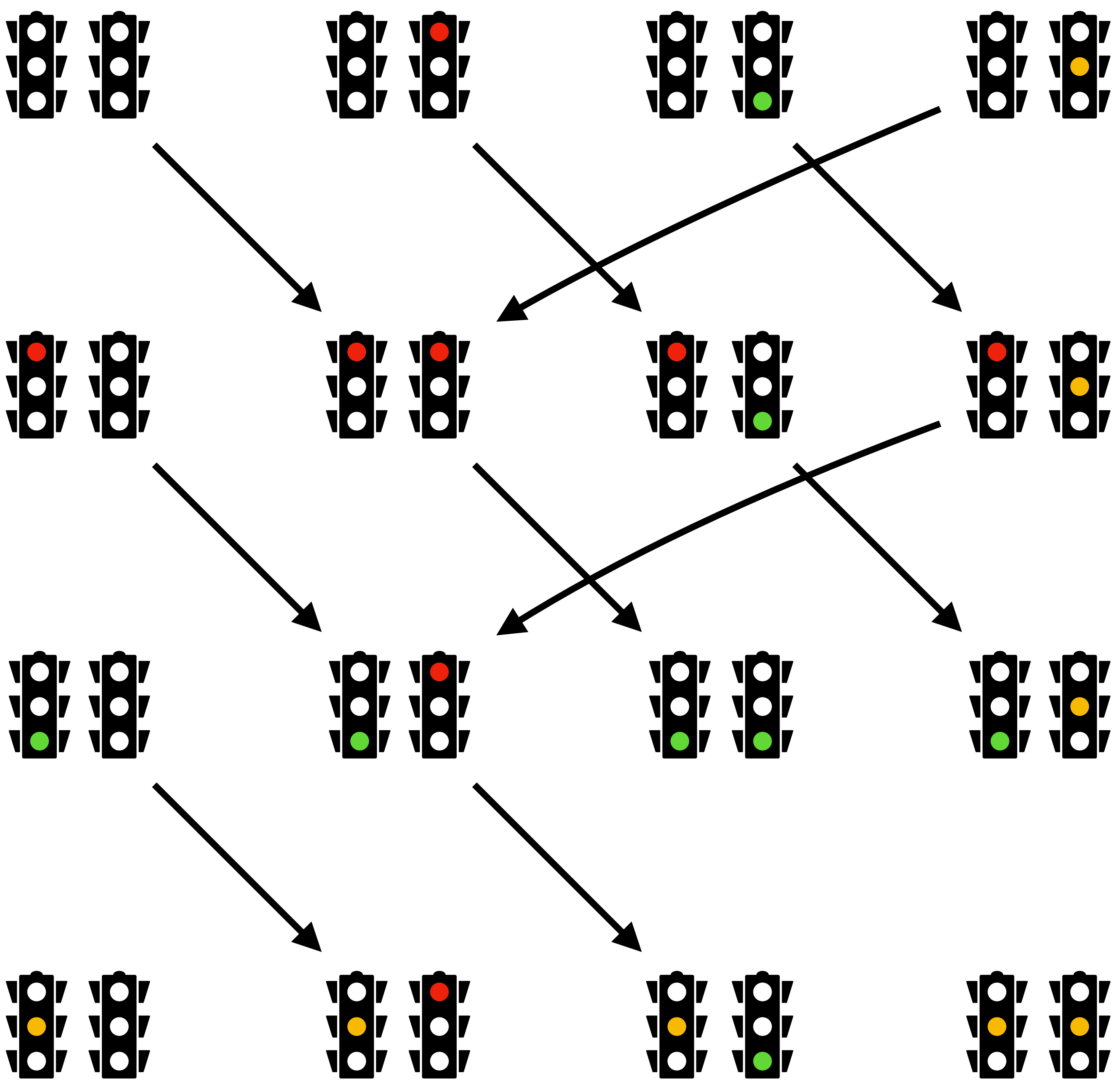

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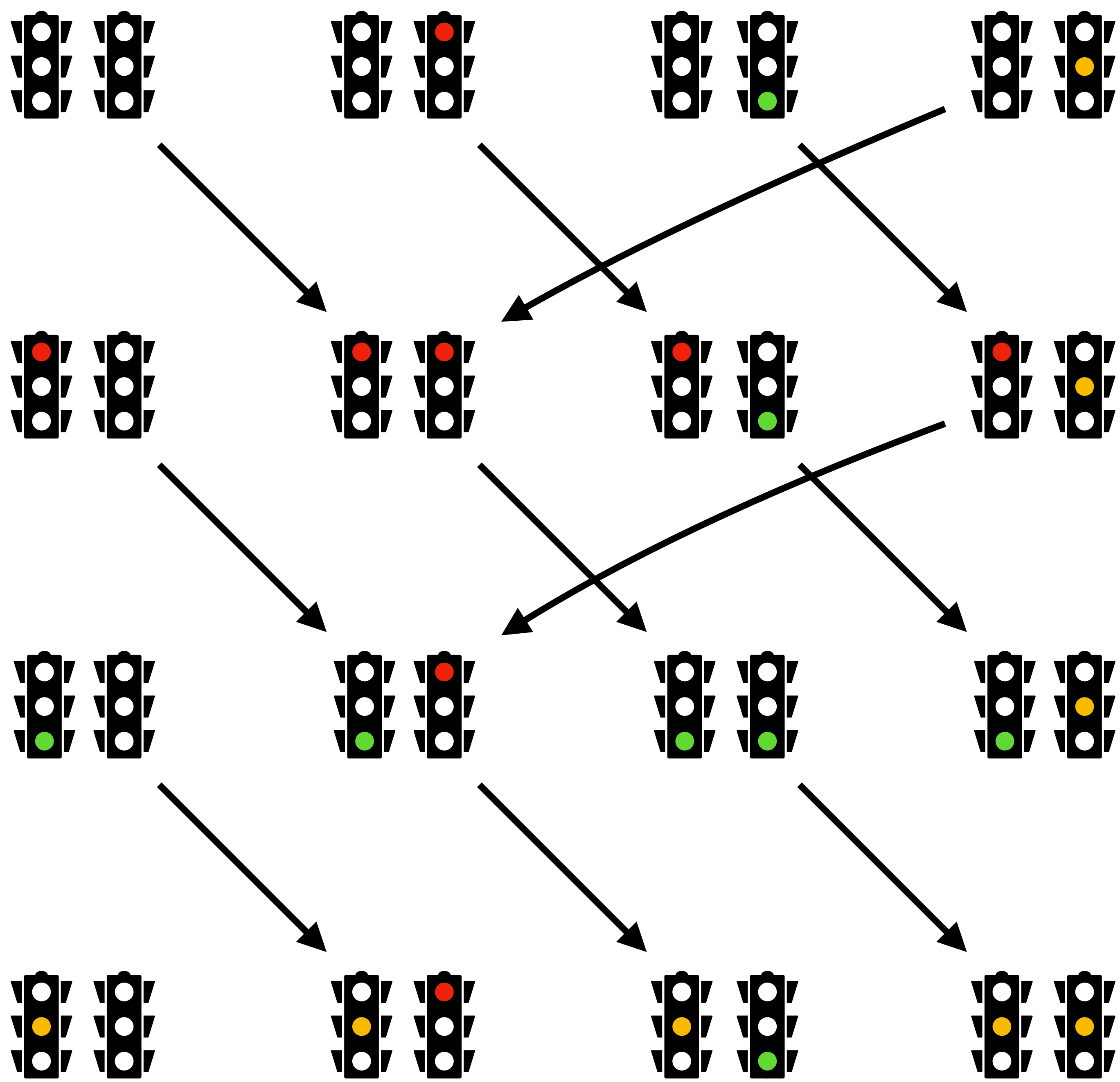

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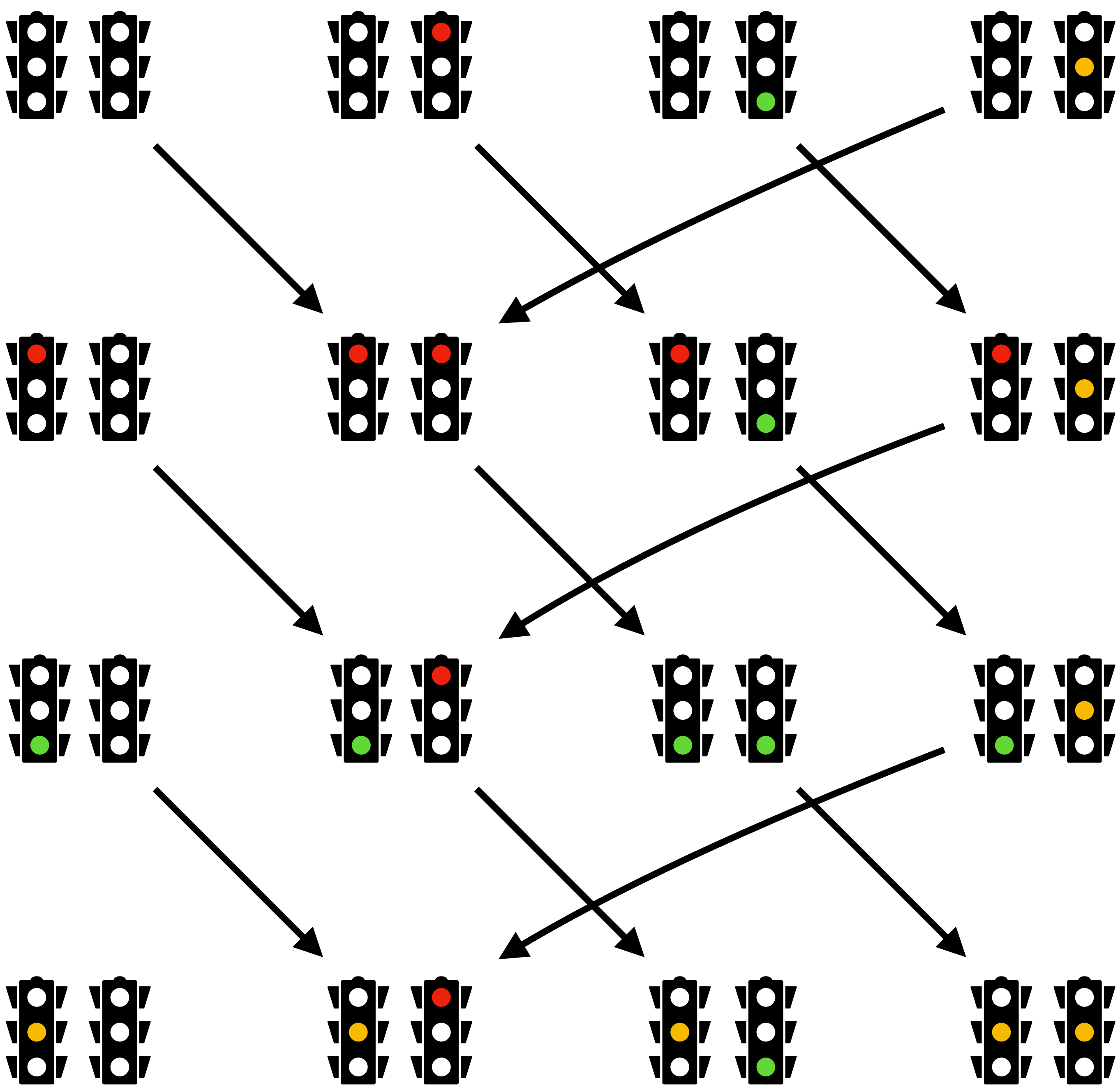

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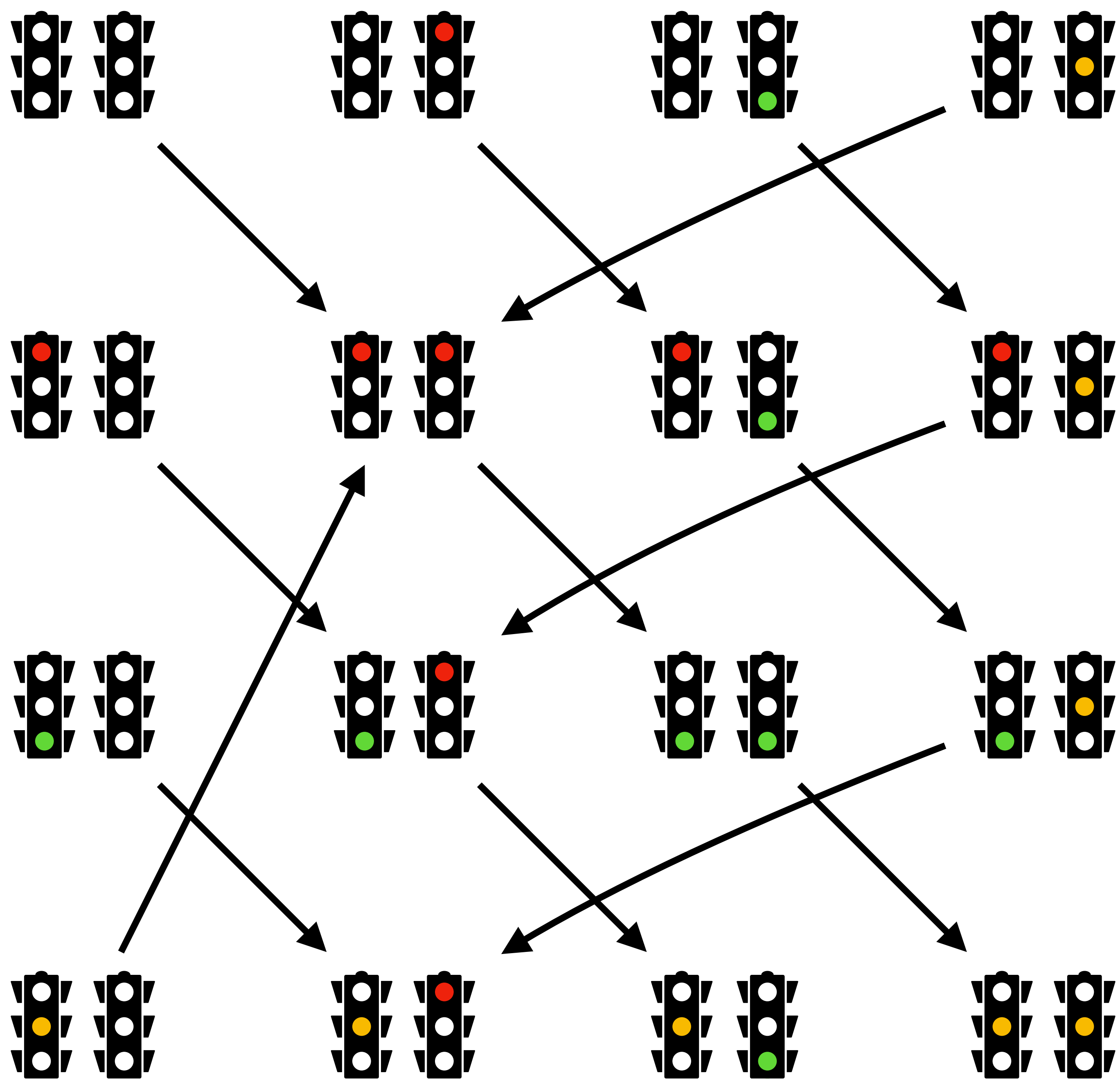

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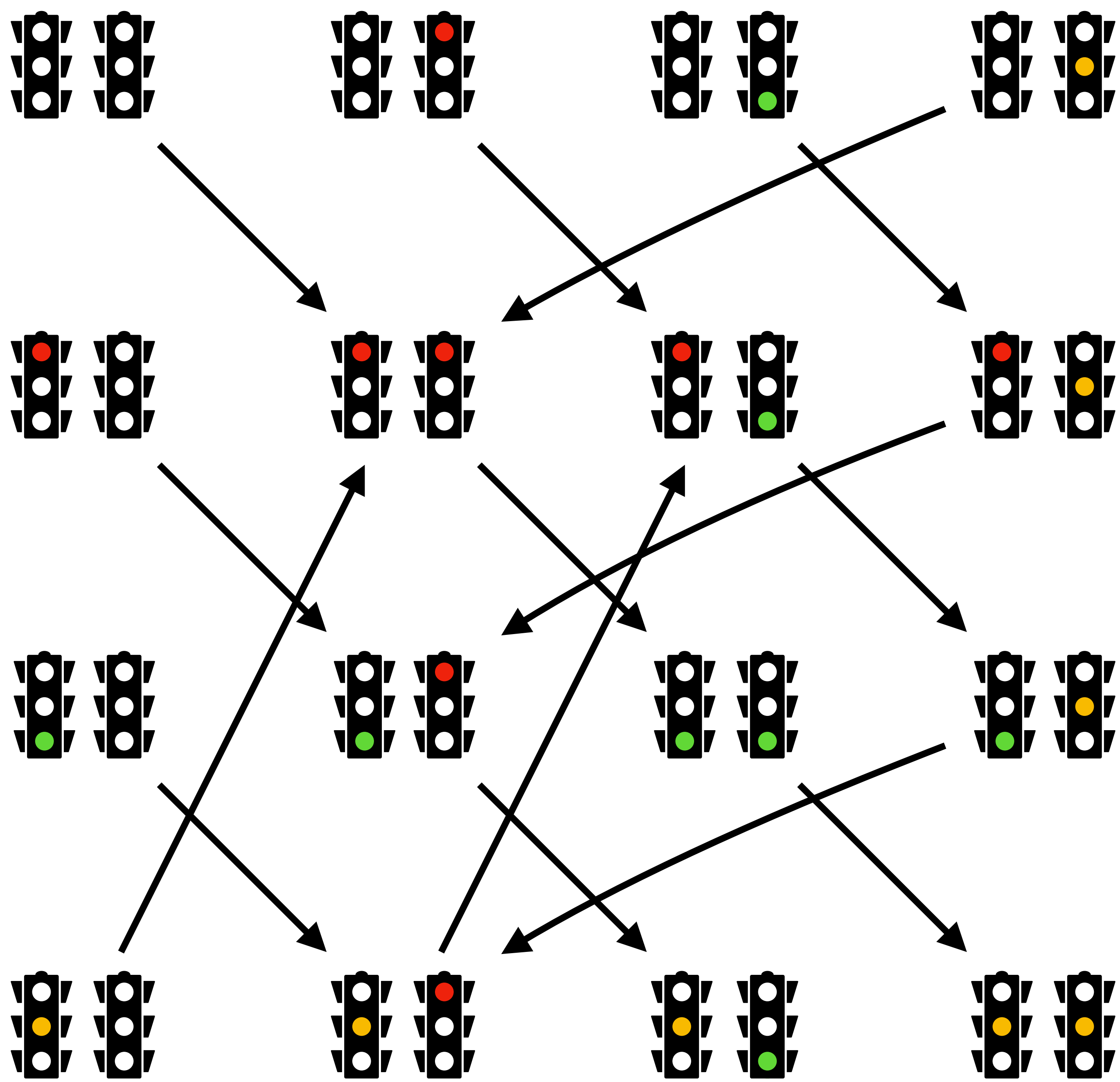

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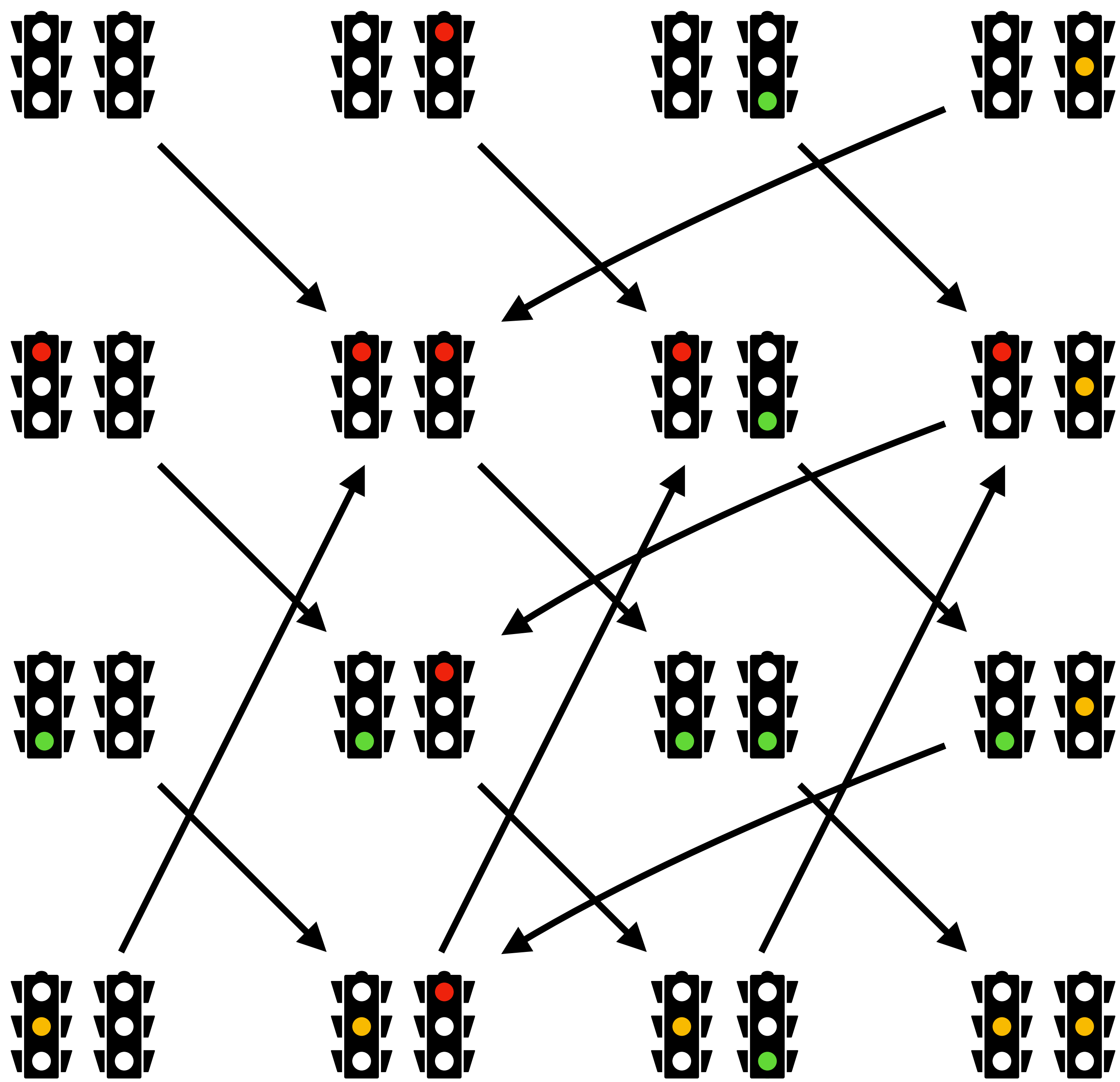

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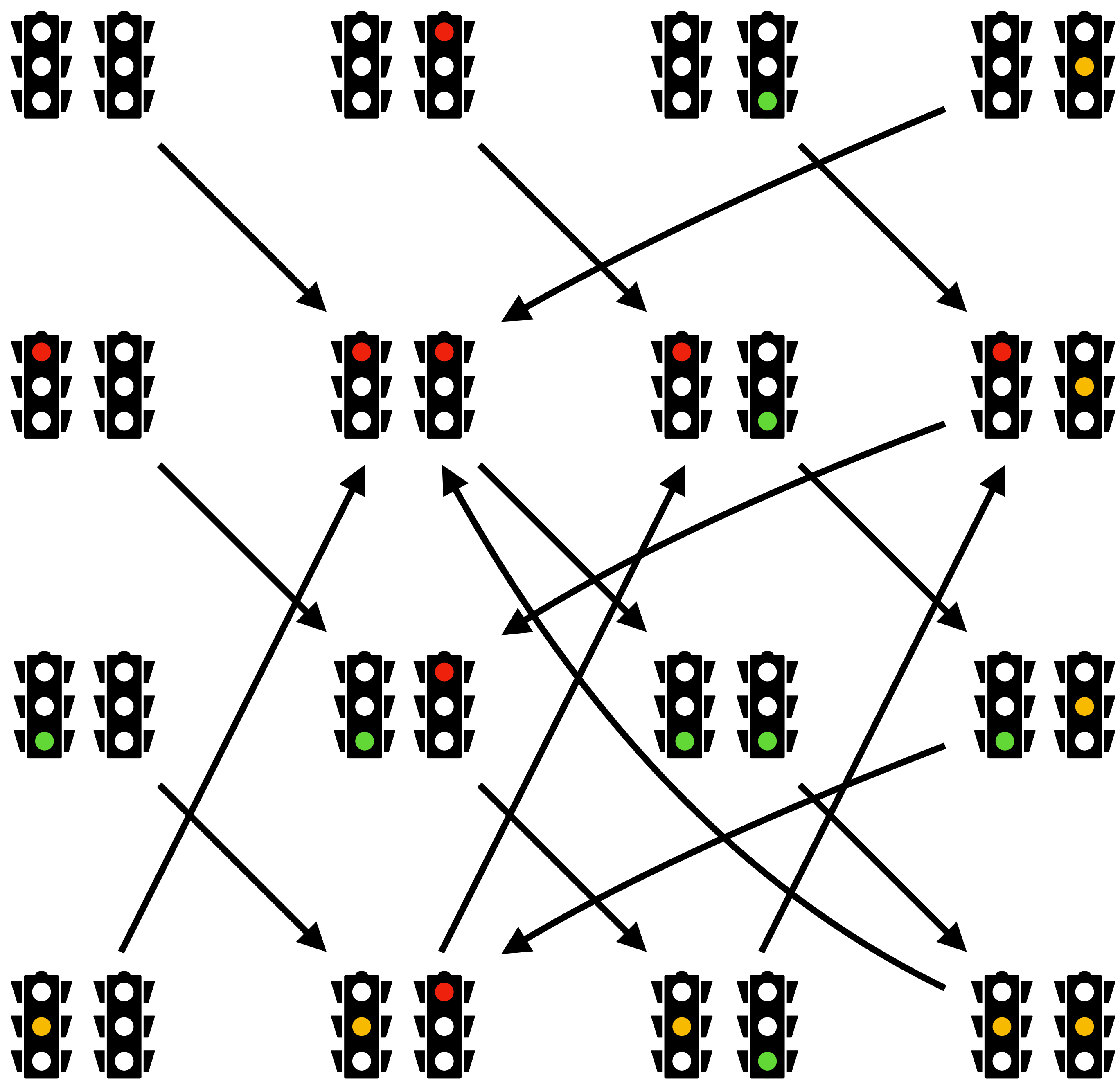

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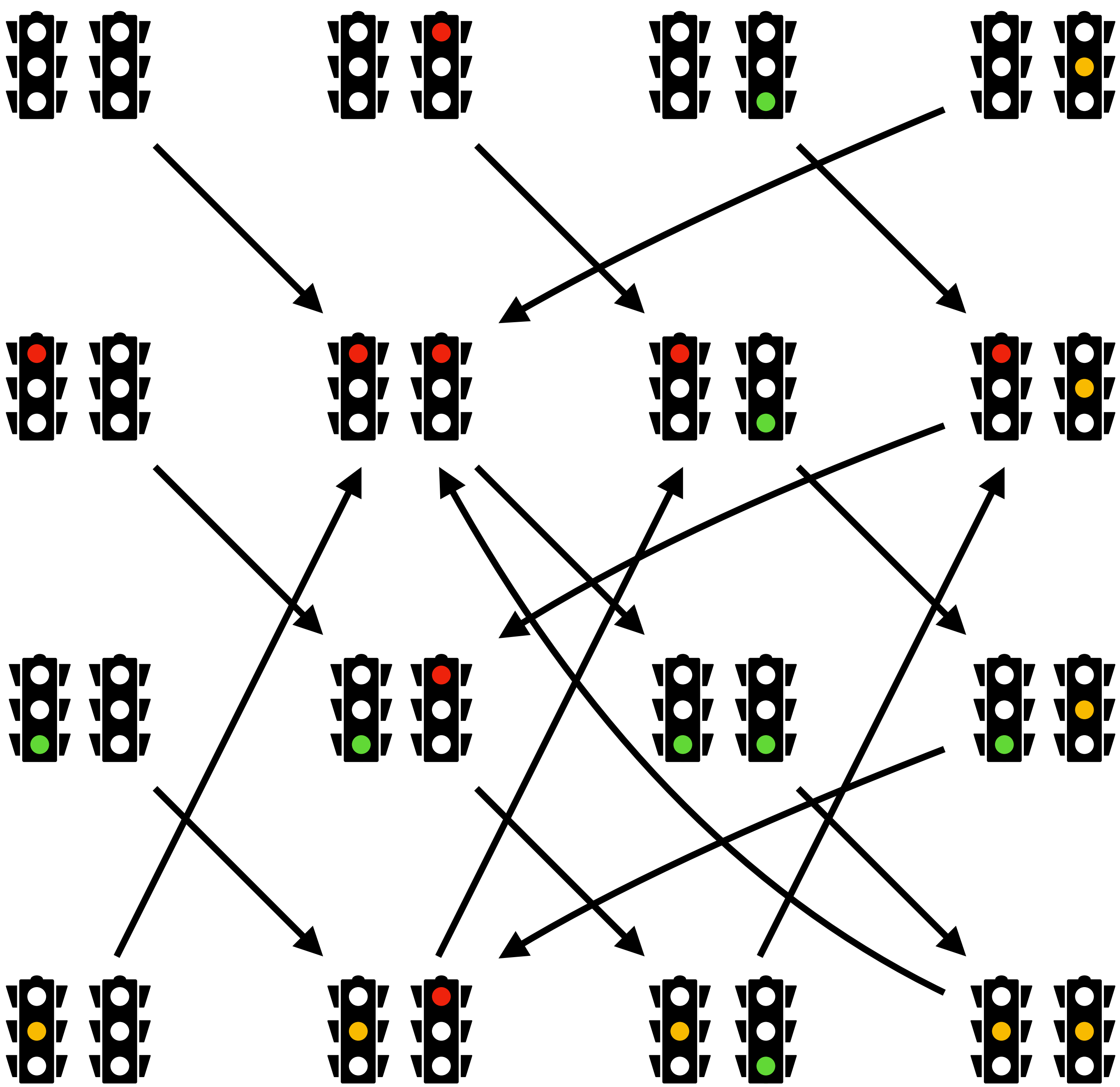

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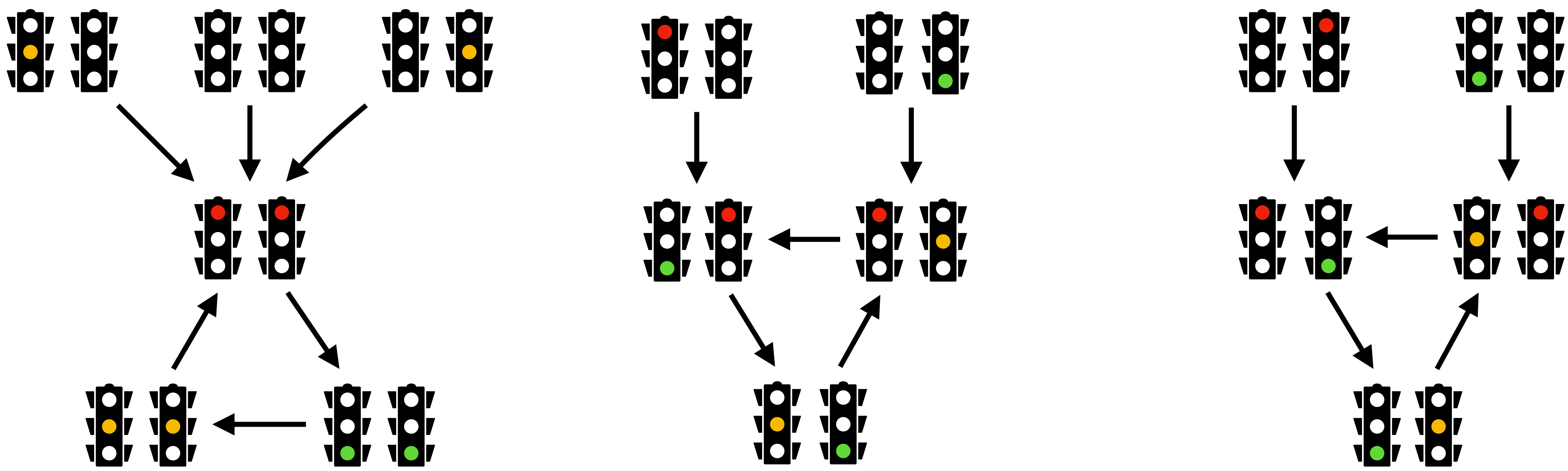

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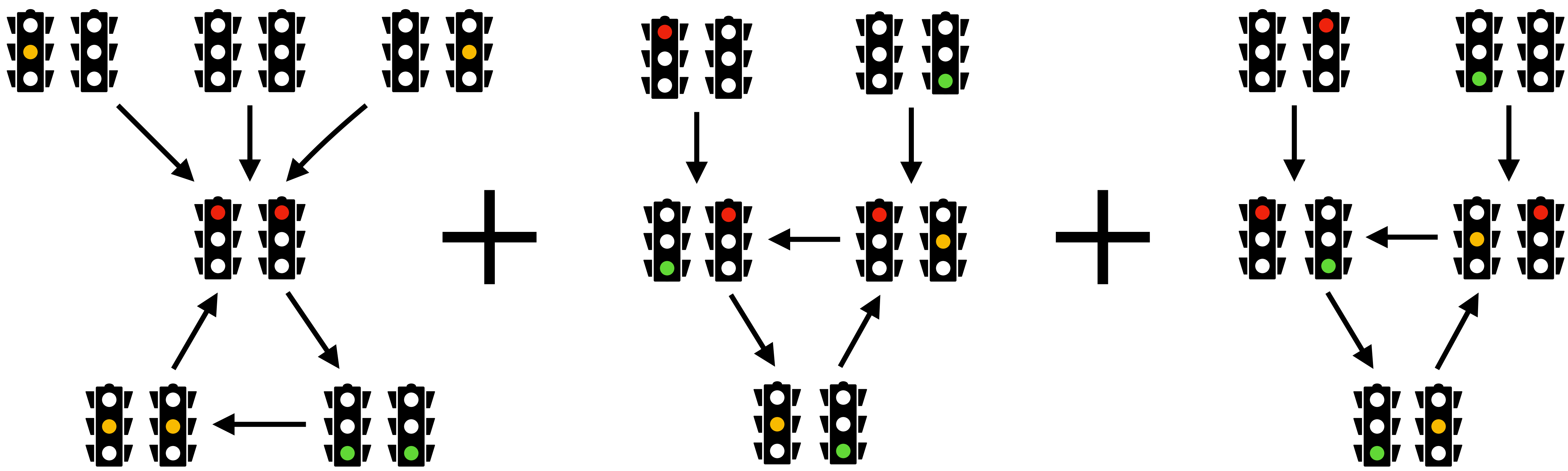
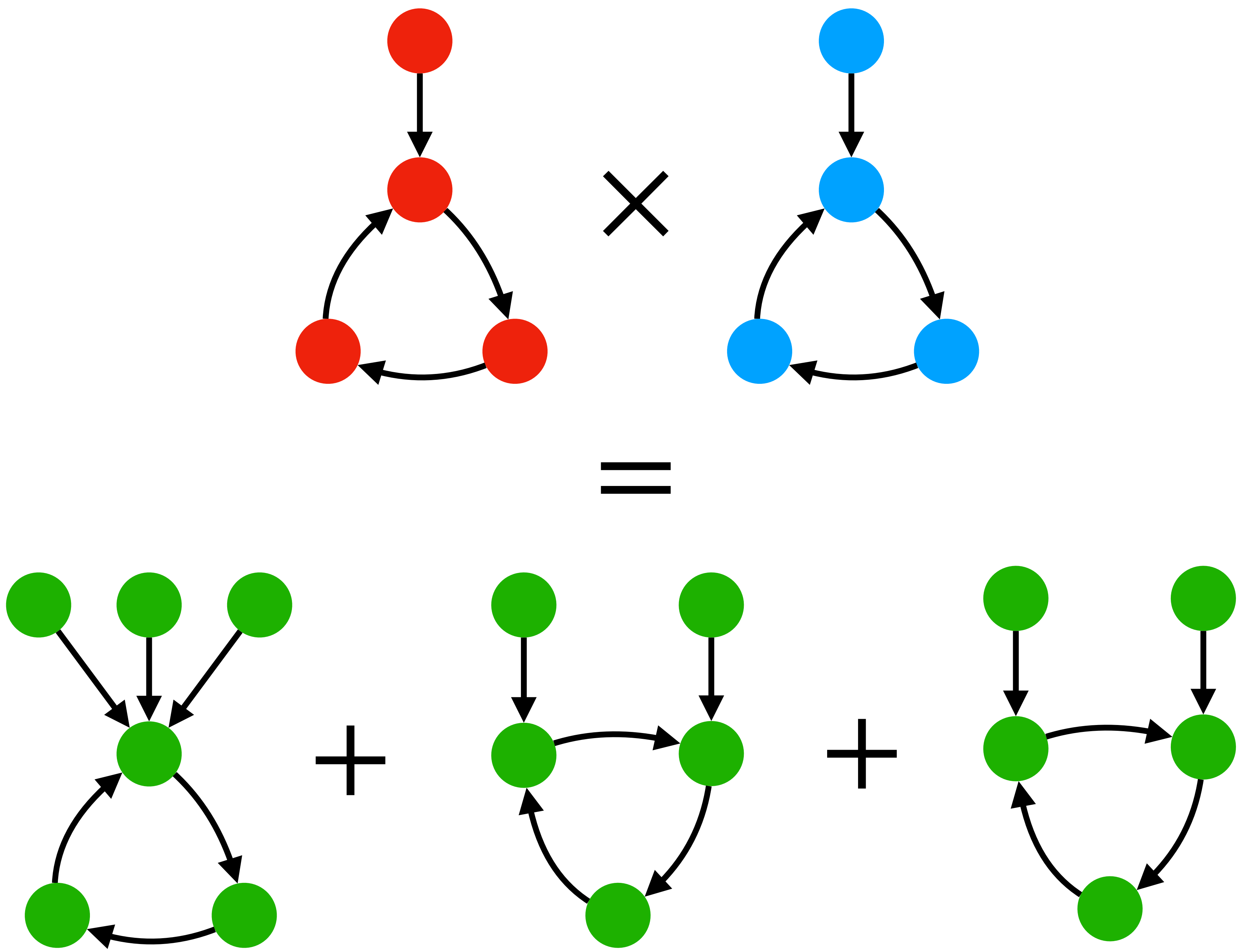


More abstractly…



$+$ and $\times$ behave as with nonnegative integers (a commutative semiring)

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- Associative: $X + (Y + Z) = (Y + X) + Z$ and $X \times (Y \times Z) = (Y \times X) \times Z$
- Neutral elements: $\varnothing + X = X$ and  $\times X = X$

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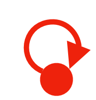
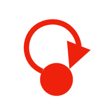
- Associative: $X + (Y + Z) = (Y + X) + Z$ and $X \times (Y \times Z) = (Y \times X) \times Z$

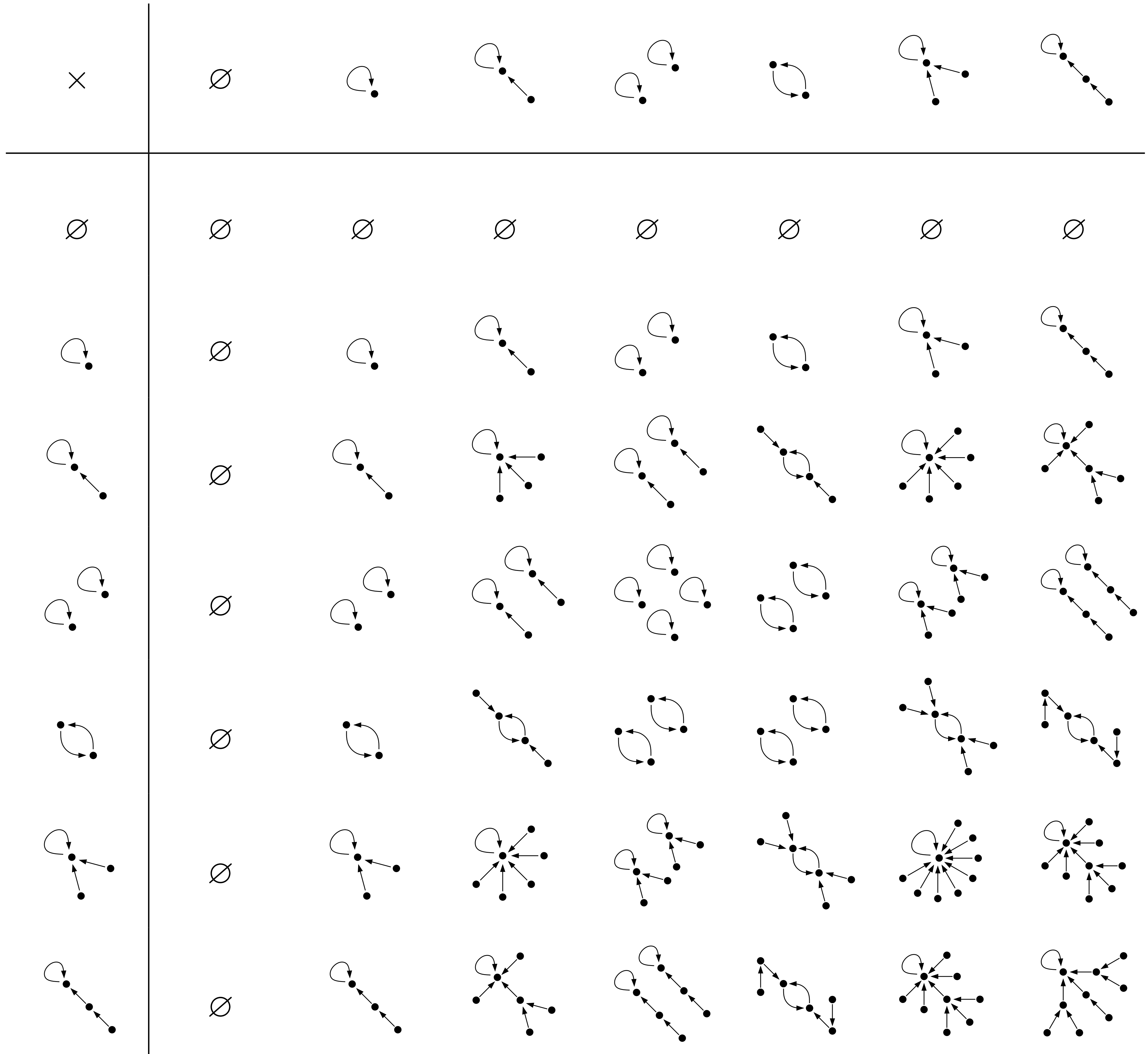
- Neutral elements: $\varnothing + X = X$ and  $\times X = X$

- Distributive: $X \times (Y + Z) = X \times Y + X \times Z$

$+$ and $\times$ behave as with nonnegative integers (a commutative semiring)

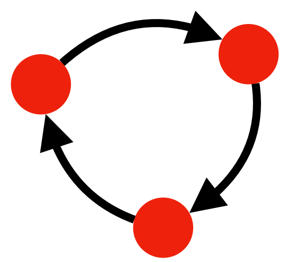
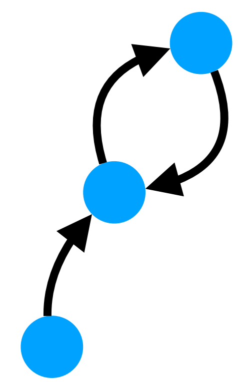
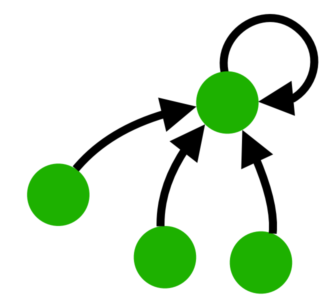
- Commutative: $X + Y = Y + X$ and $X \times Y = Y \times X$
- Associative: $X + (Y + Z) = (Y + X) + Z$ and $X \times (Y \times Z) = (Y \times X) \times Z$
- Neutral elements: $\varnothing + X = X$ and  $\times X = X$
- Distributive: $X \times (Y + Z) = X \times Y + X \times Z$
- Multiplication by zero: $\varnothing \times X = \varnothing$

Multiplication table

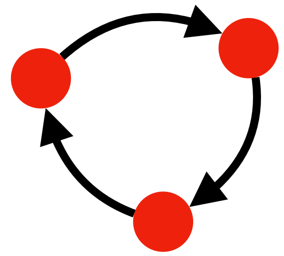
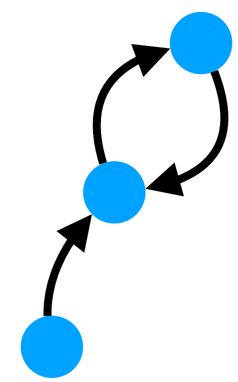
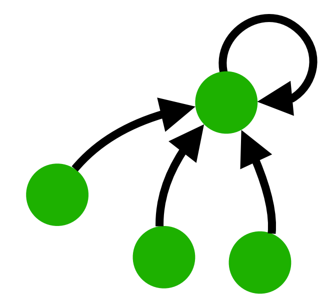


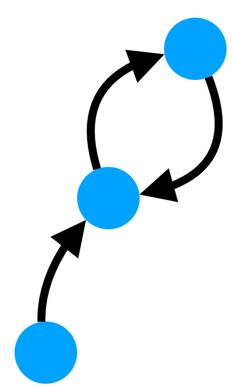
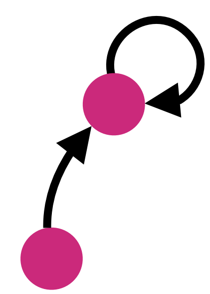
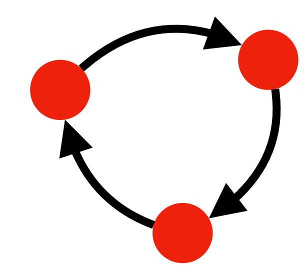
Equations for decomposing systems

Eqns over dynamical systems

 $X + Y^2 =$  $Z +$ 

Eqns over dynamical systems

 $X + Y^2 =$  $Z +$ 

$X =$  $Y =$  $Z =$ 

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- We are still unsure if equations in one single variable, like $AX = B$, can be solved efficiently (conjecture: no)

Identifying basic building blocks

Scenario

DynaSys Inc.

Scenario

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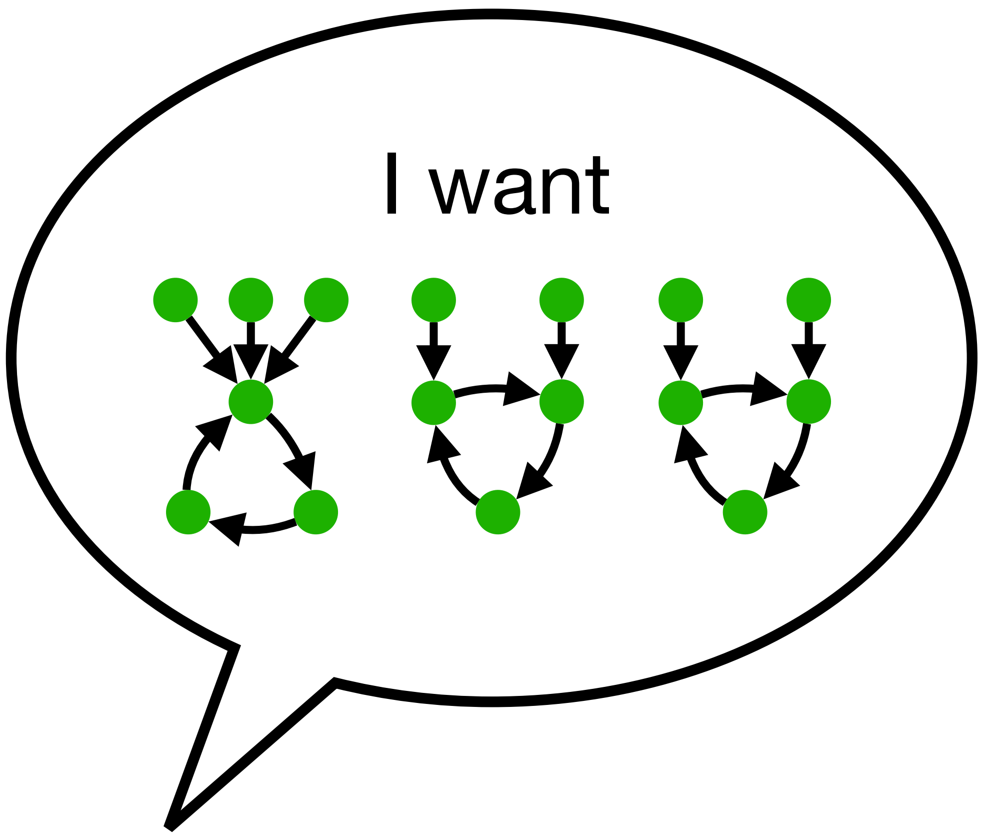


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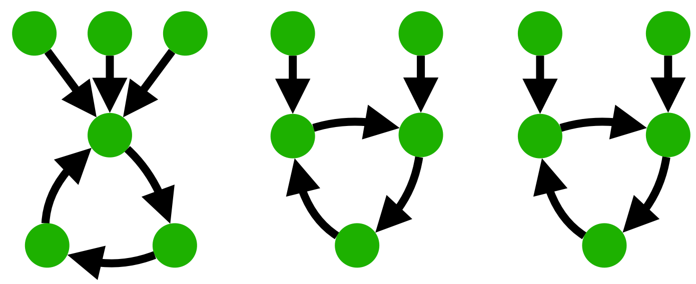


$$= X \times Y + Z$$

DynaSys Inc.

Scenario

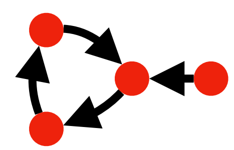
$$X =$$

$$Y =$$

$$Z =$$


DynaSys Inc.

Scenario

$X =$ 

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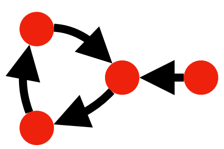
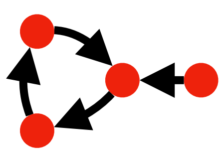
DynaSys Inc.

Scenario

$X =$ 

$Y =$ 

$Z =$

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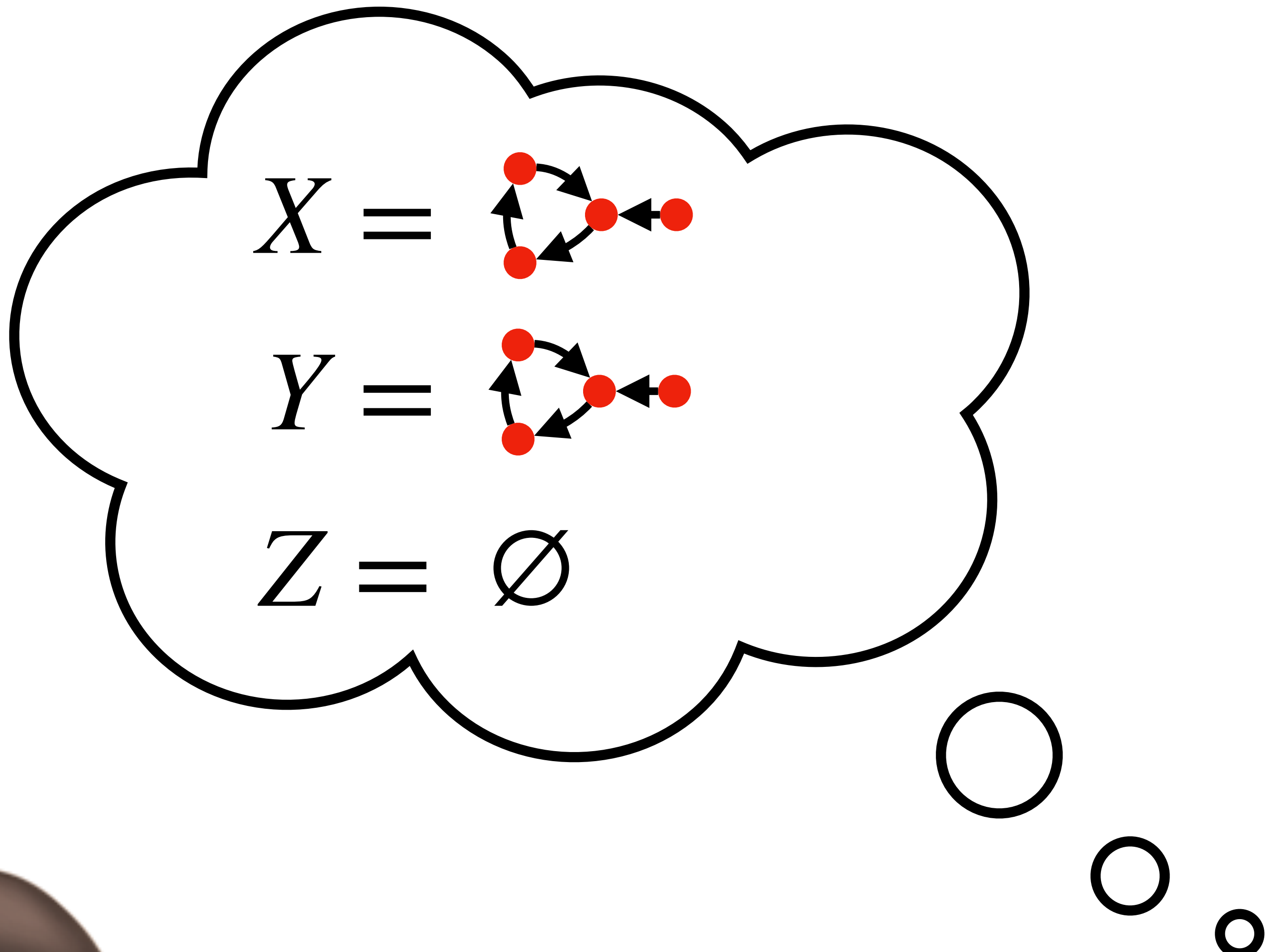

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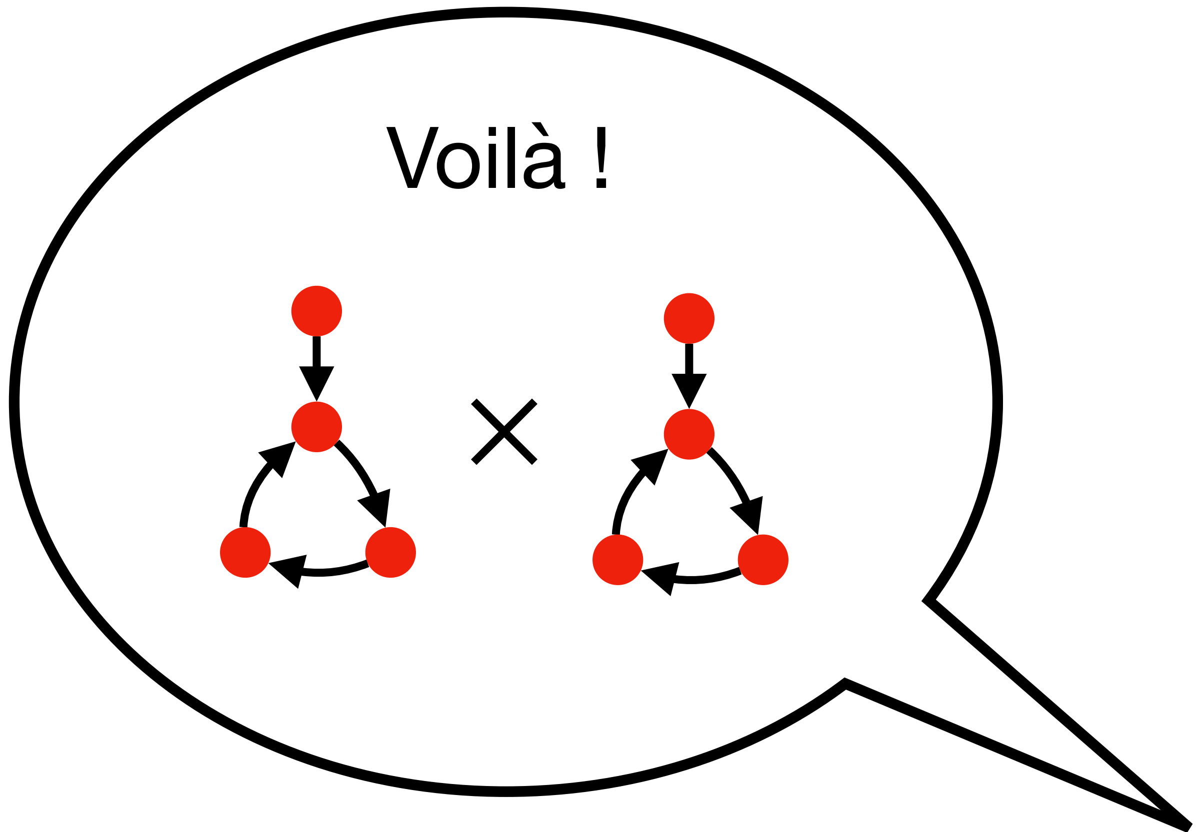

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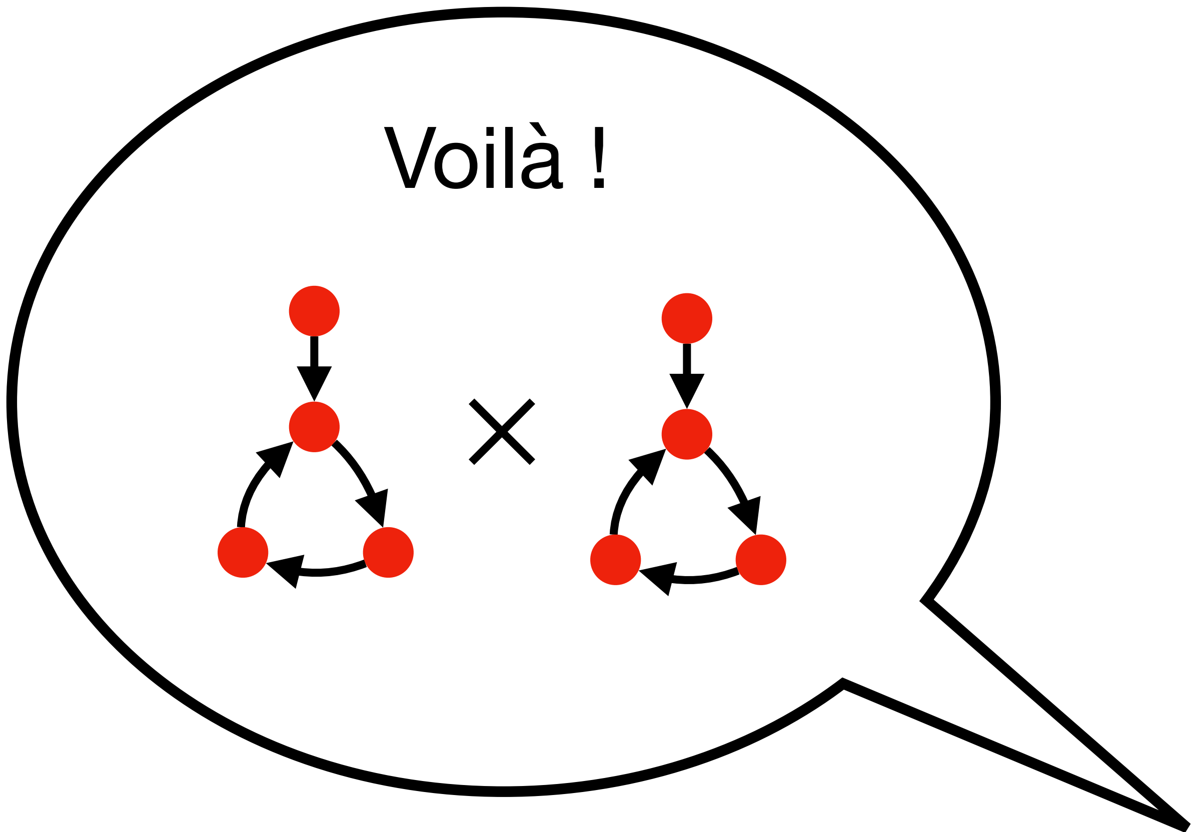
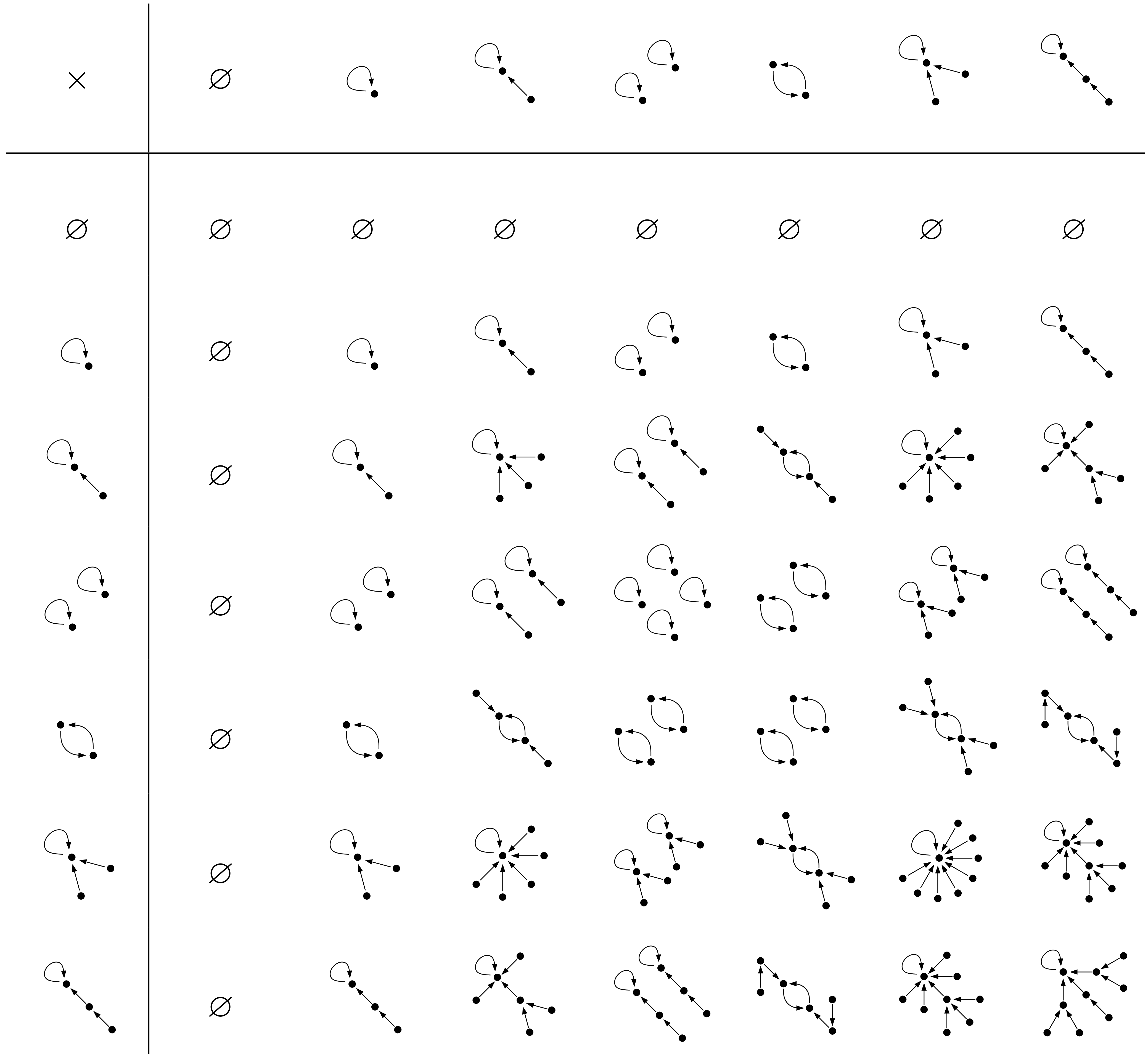
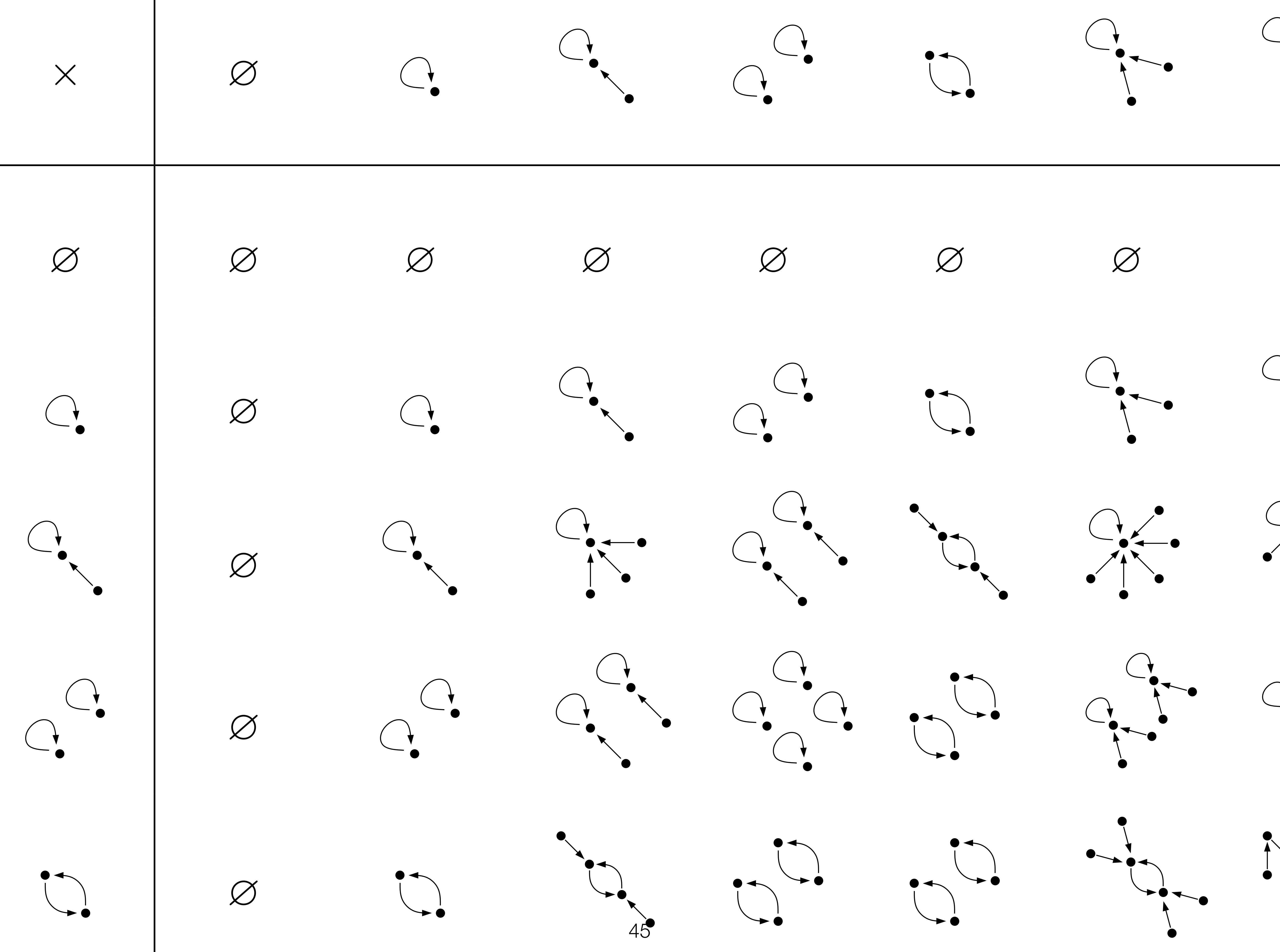
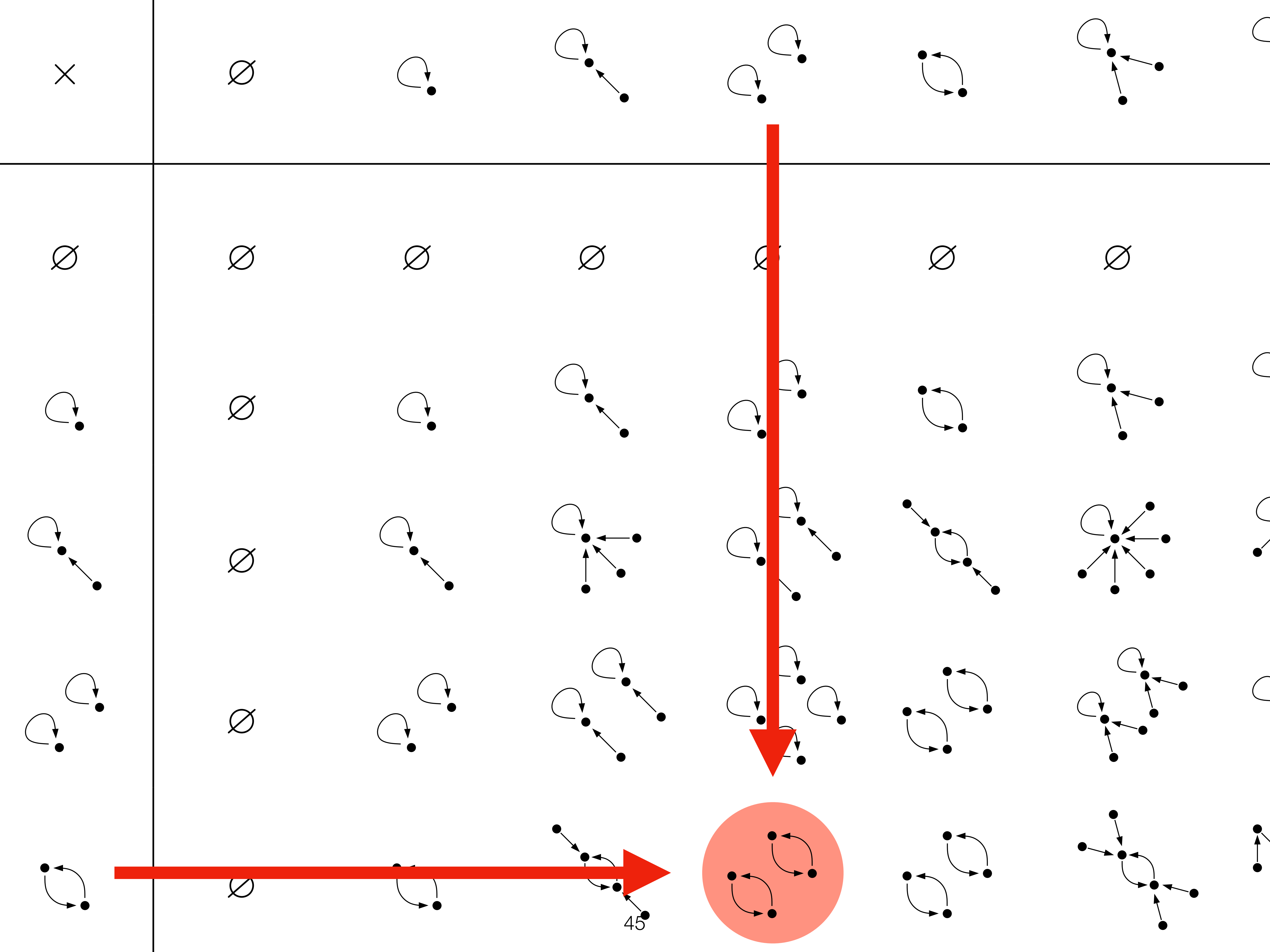


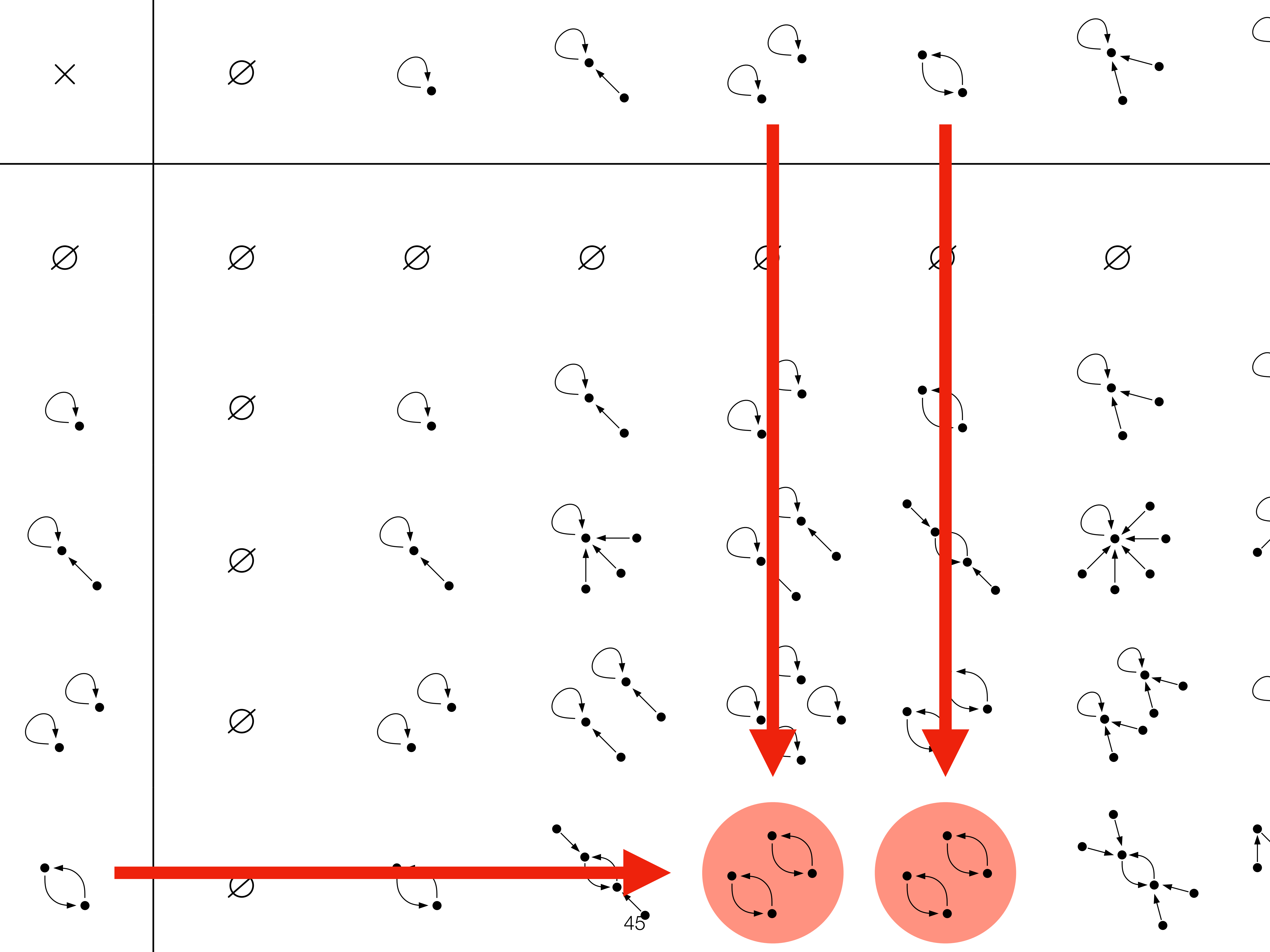
DynaSys Inc.

No unique factorisation into irreducibles!

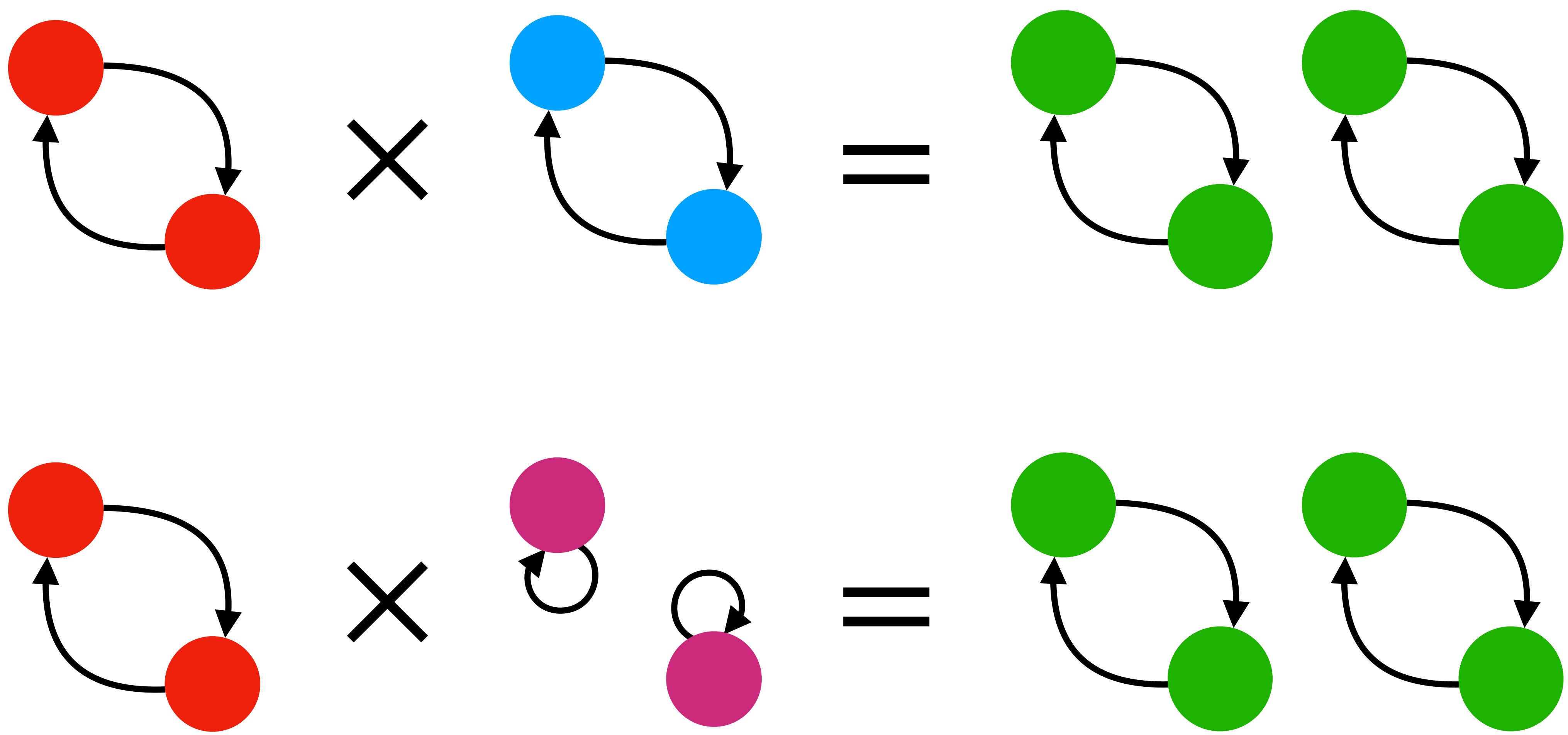








Multiple factorisations



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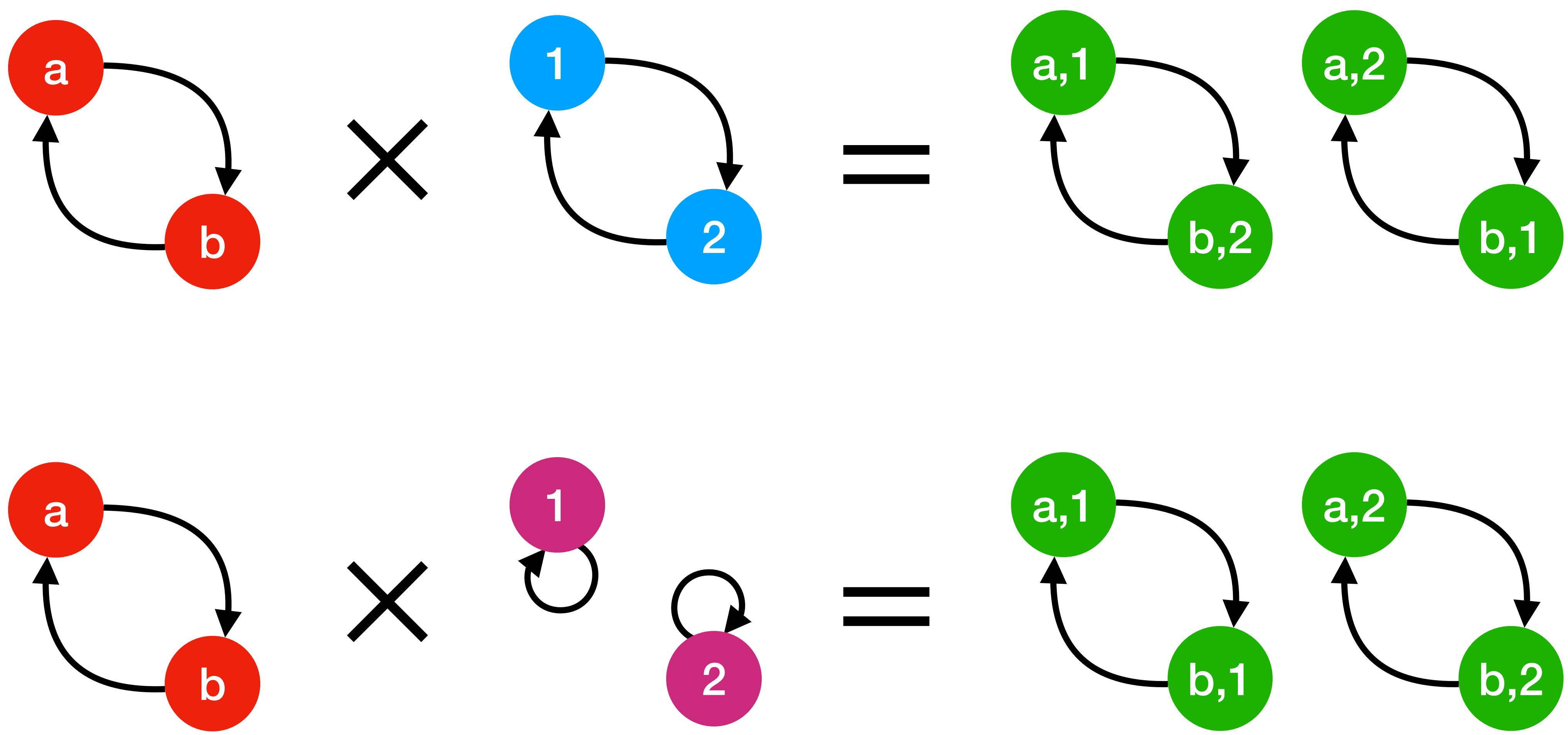

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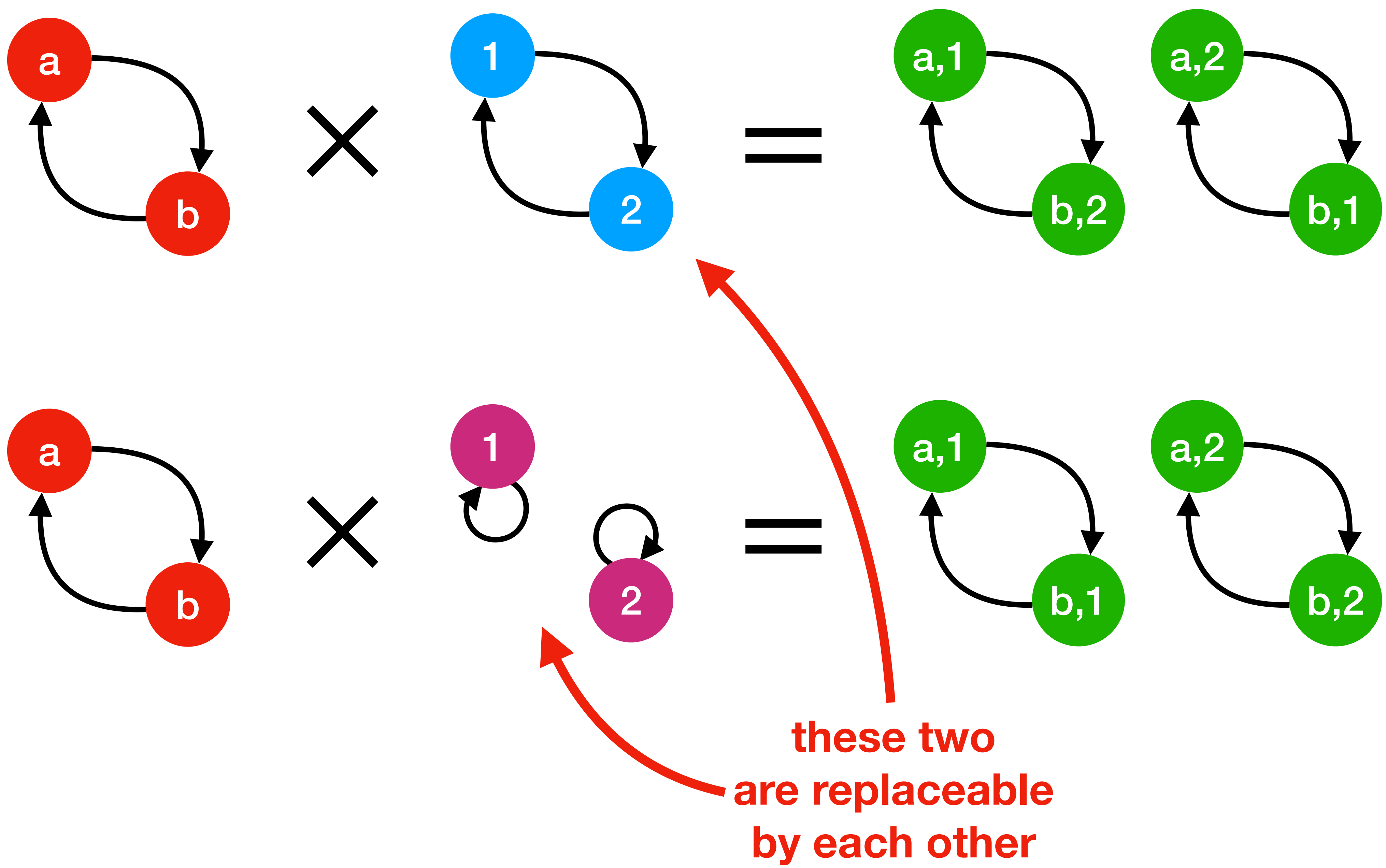

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- Decompose our system A as a product of smaller, irreducible systems $A = A_1 \times A_2 \times \cdots \times A_n$
- But in general there are multiple ways of doing that: $A = B_1 \times B_2 \times \cdots \times B_m$
- So $A_1, A_2, \ldots, A_n$ might all be replaceable

Prime systems

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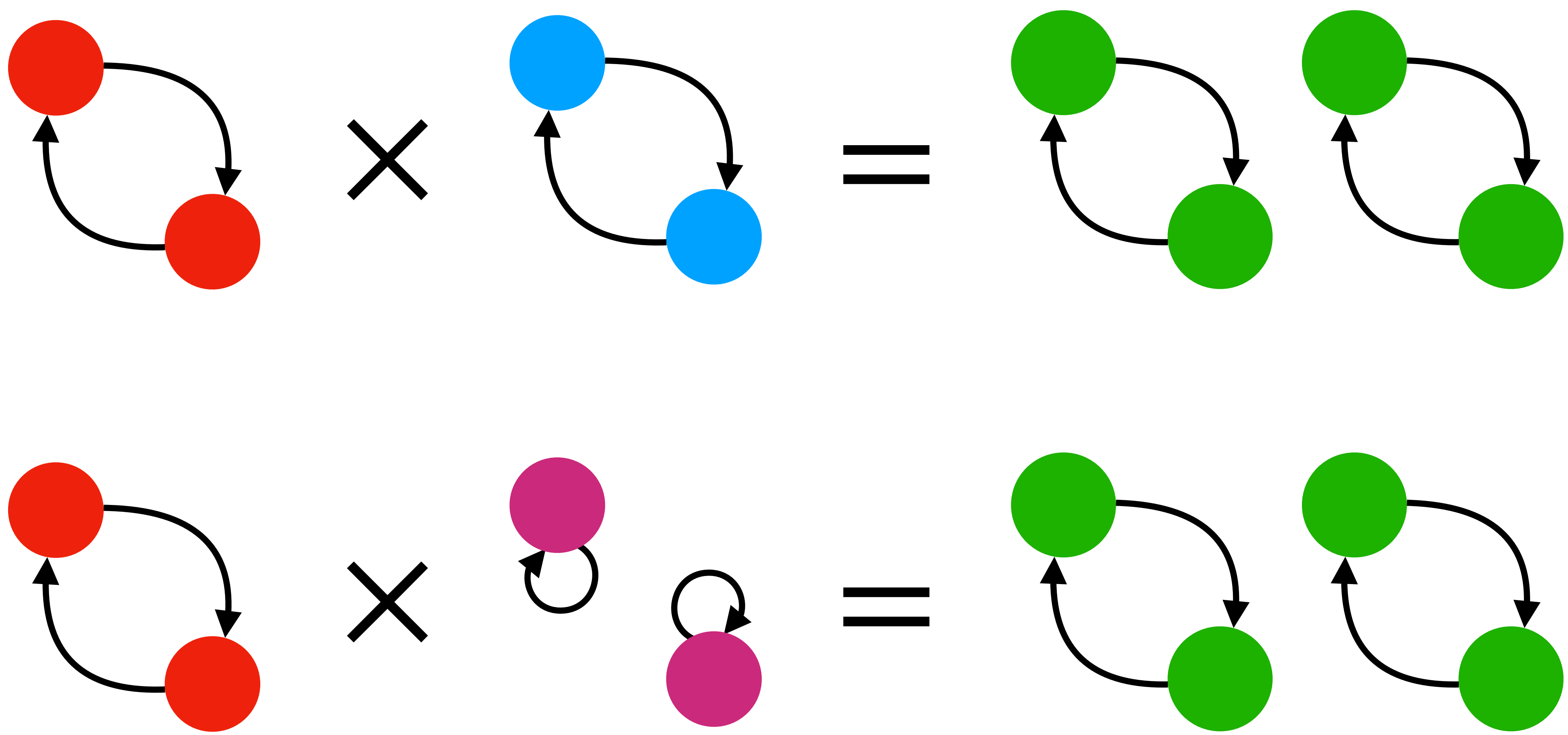
- If a prime appears in one factorisation of a system, then it appears in all the others as well (it is irreplaceable)
- So we want at least all primes in our warehouse (even though that's not enough…)

Primes are not sufficient



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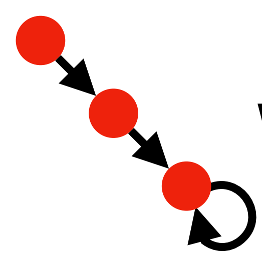
- This guy here?  We don't know if it's prime or not!

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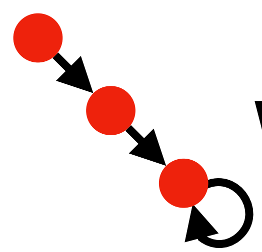
- This guy here?  We don't know if it's prime or not!
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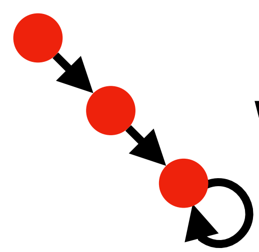
- This guy here?  We don't know if it's prime or not!
- A counterexample to the primality of P is two systems A, B such that P divides $A \times B$ but neither A nor B
- Those A and B can be much bigger than P,
  but we don't know how much, so no algorithm yet…

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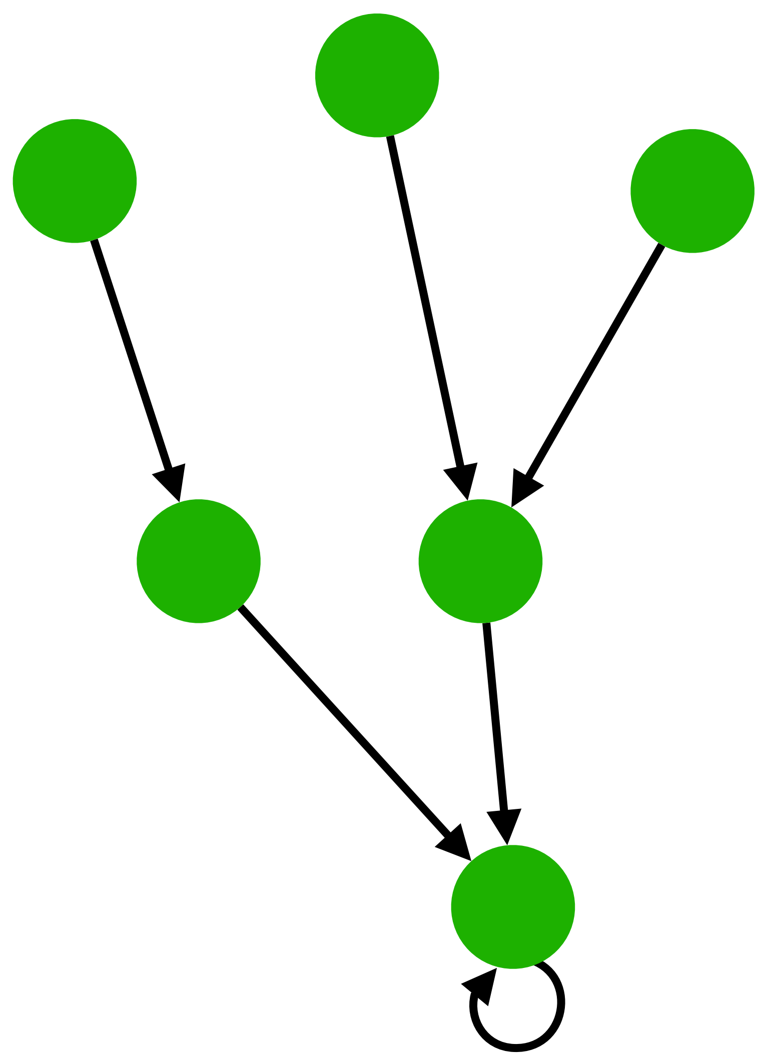

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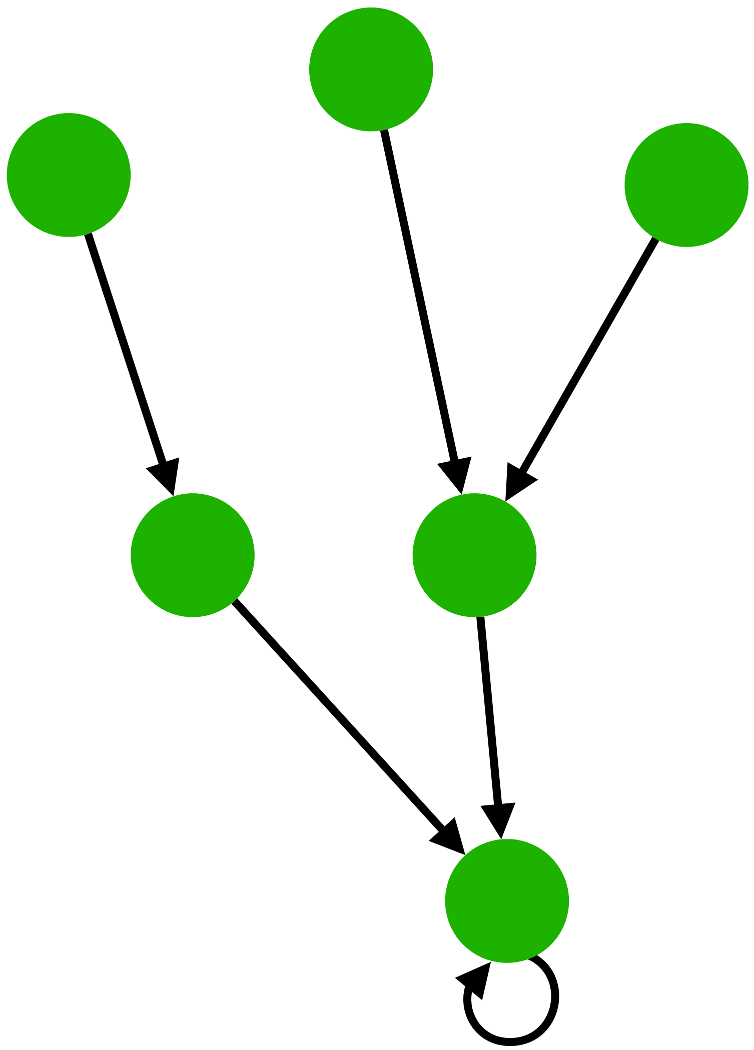

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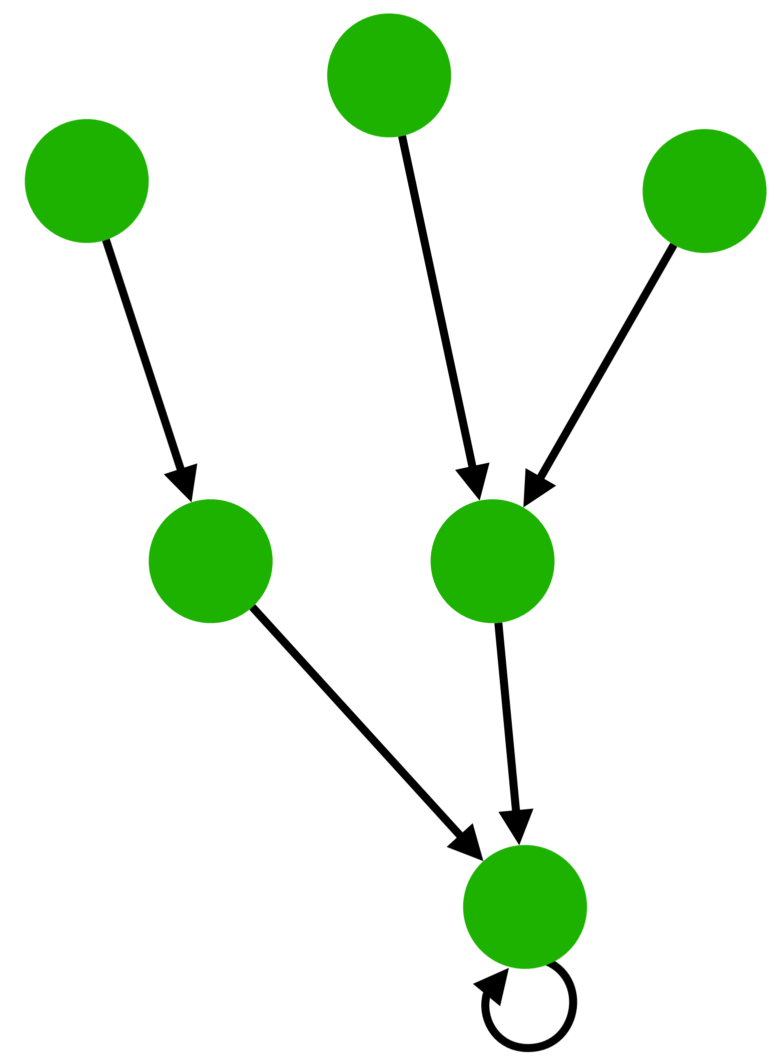

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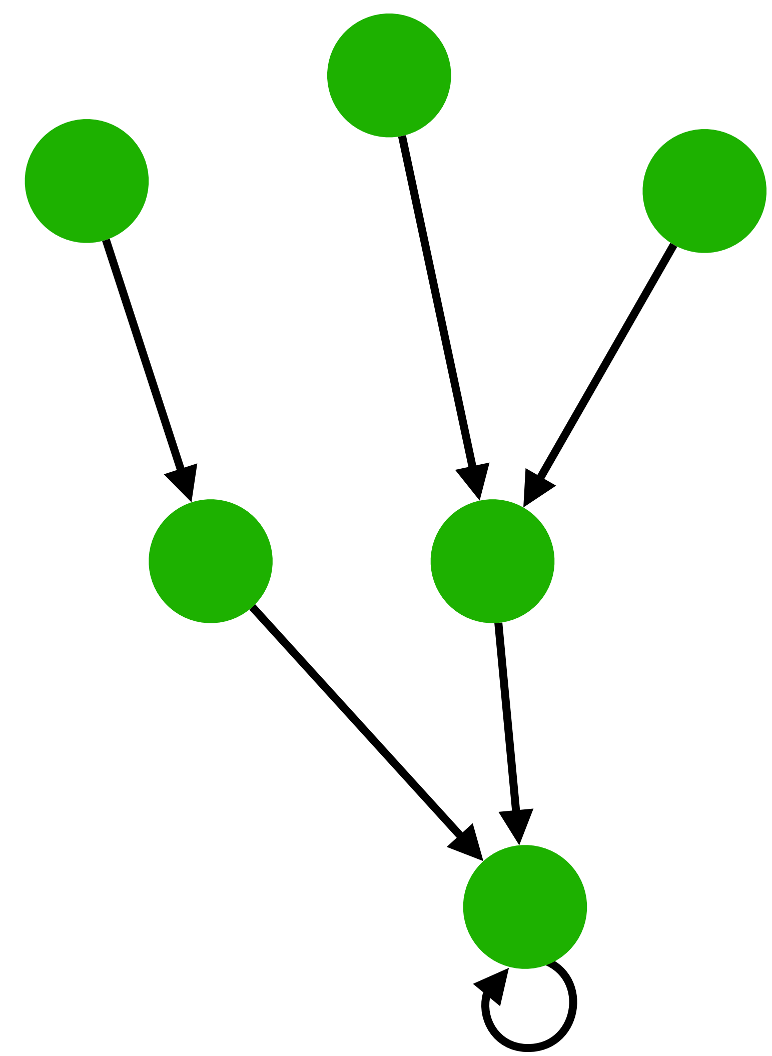
- Connected
- Fixed point (no cycles of length > 1)
- gcd of the number of predecessors across all states must be 1



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- An enumeration algorithm for dynamical systems up to isomorphism, to find examples and counterexamples (almost done)

- Find out if prime systems exist, or at least find a primality algorithm!

Bibliography

- A. Dennunzio, V. Dorigatti, E. Formenti, L. Manzoni, A.E. Porreca, Polynomial equations over finite, discrete-time dynamical systems

- F. Doré, E. Formenti, A.E. Porreca, S. Riva, Algorithmic reconstruction of discrete dynamics (and its bibliography)

- É. Naquin, M. Gadouleau, Factorisation in the semiring of finite dynamical systems (new and fresh)

- A.E. Porreca, E. Timofeeva, Efficient generation of functional digraphs up to isomorphism (soon 🤞)

Thanks for your attention!

Any questions?