

# Dynamical systems and their algebra

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# Main idea

# Main idea

**“If you liked it then you shoulda put a semiring on it”**

Beyoncé Knowles, “Single Ladies (Put a Semiring on It)”

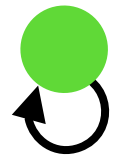
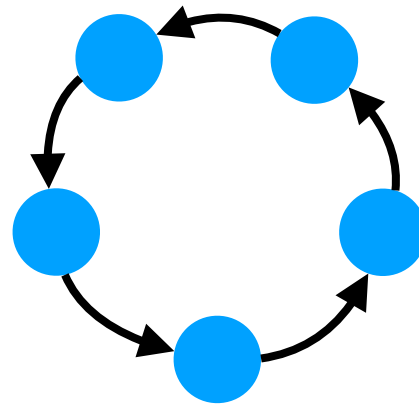
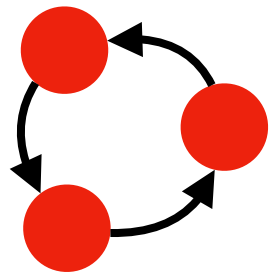
In: Beyoncé Knowles, Mathew Knowles (exec. prod.)

*I Am... Sasha Fierce*, Columbia Records, 2008

<https://youtu.be/4m1EFMoRFvY>

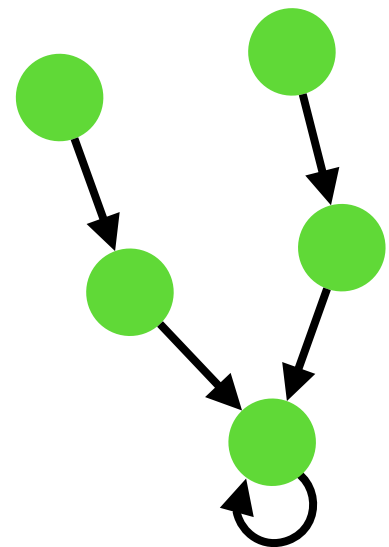
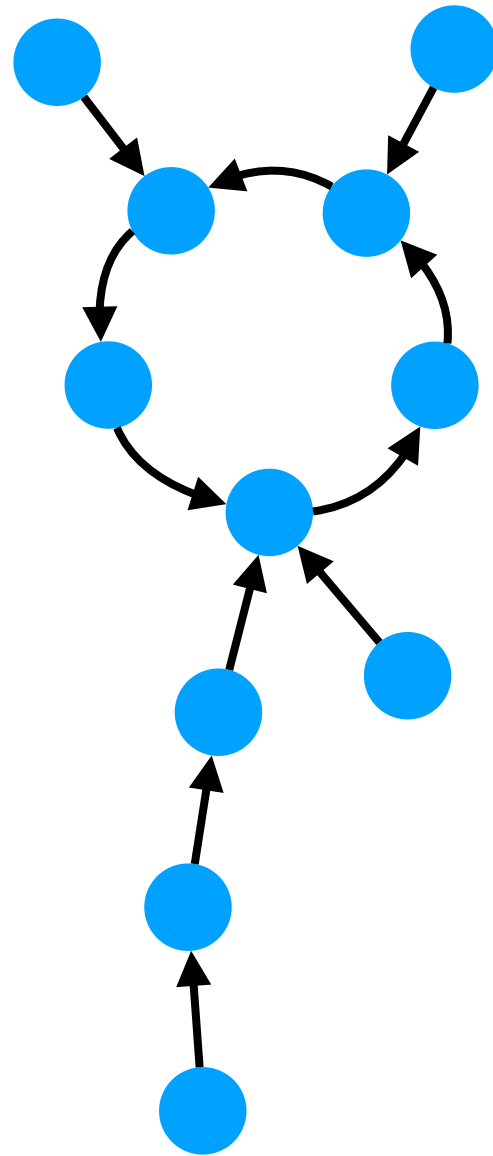
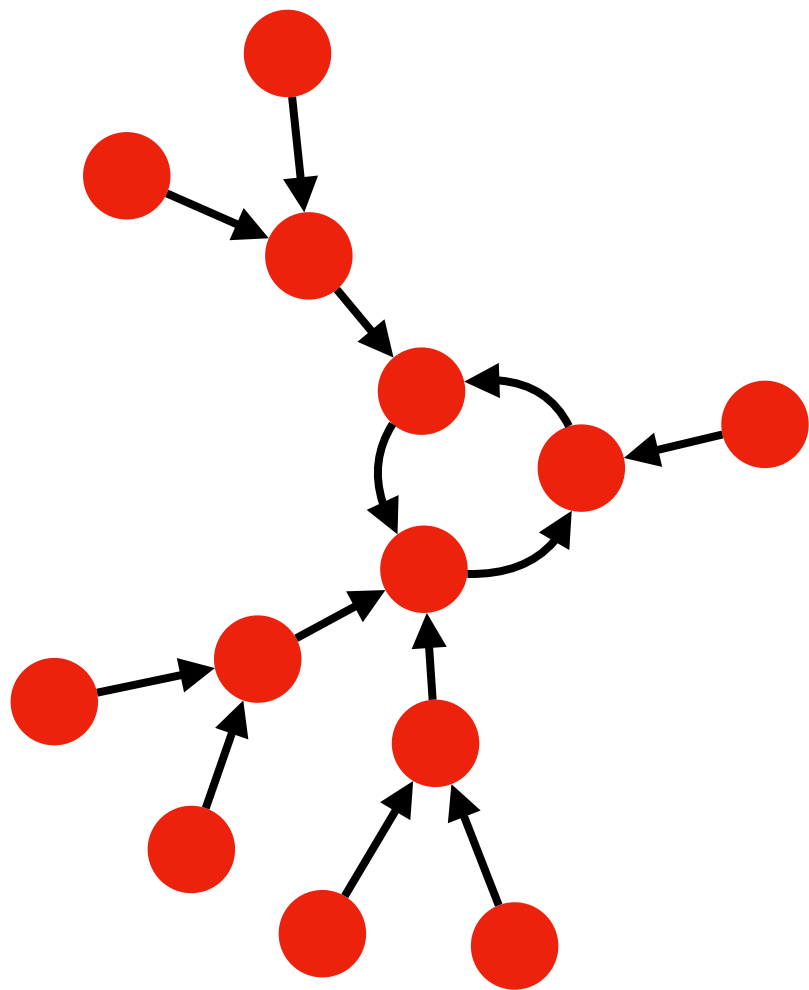
# General shape of a dynamical system

A few limit cycles



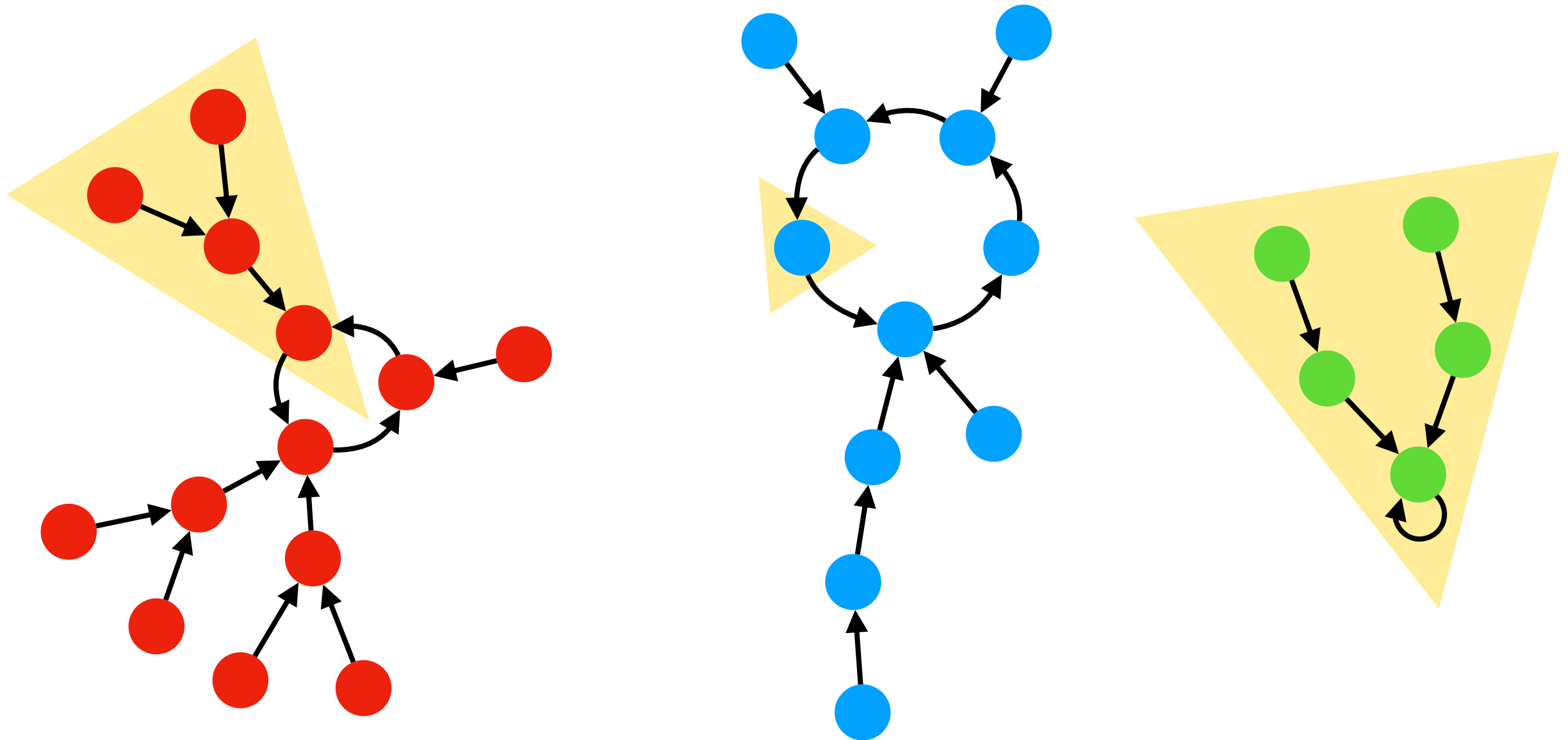
# General shape of a dynamical system

A few limit cycles **with trees going in**



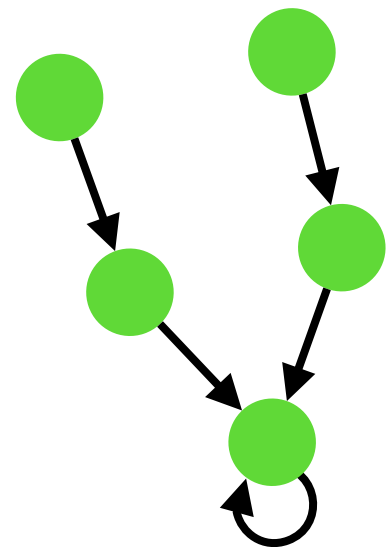
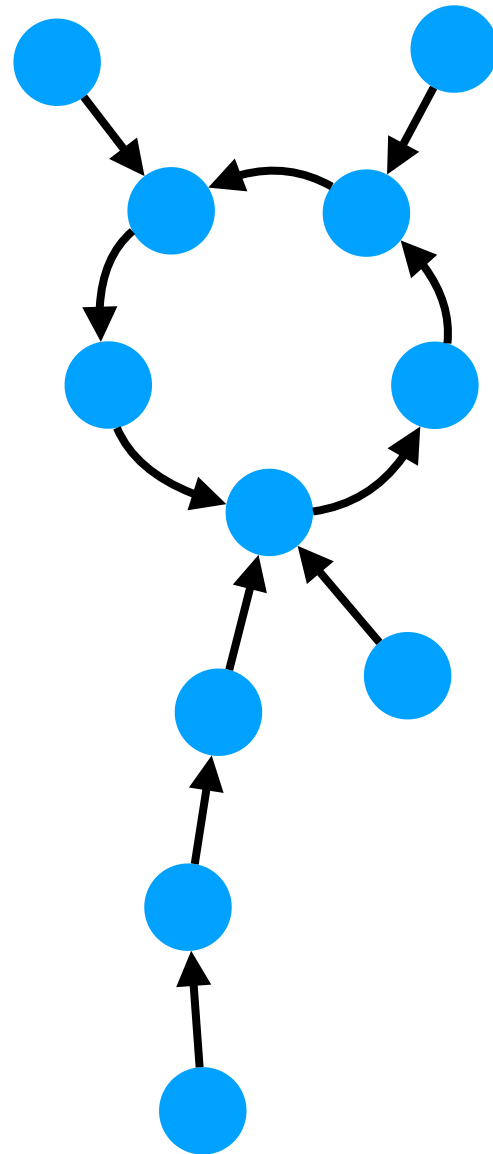
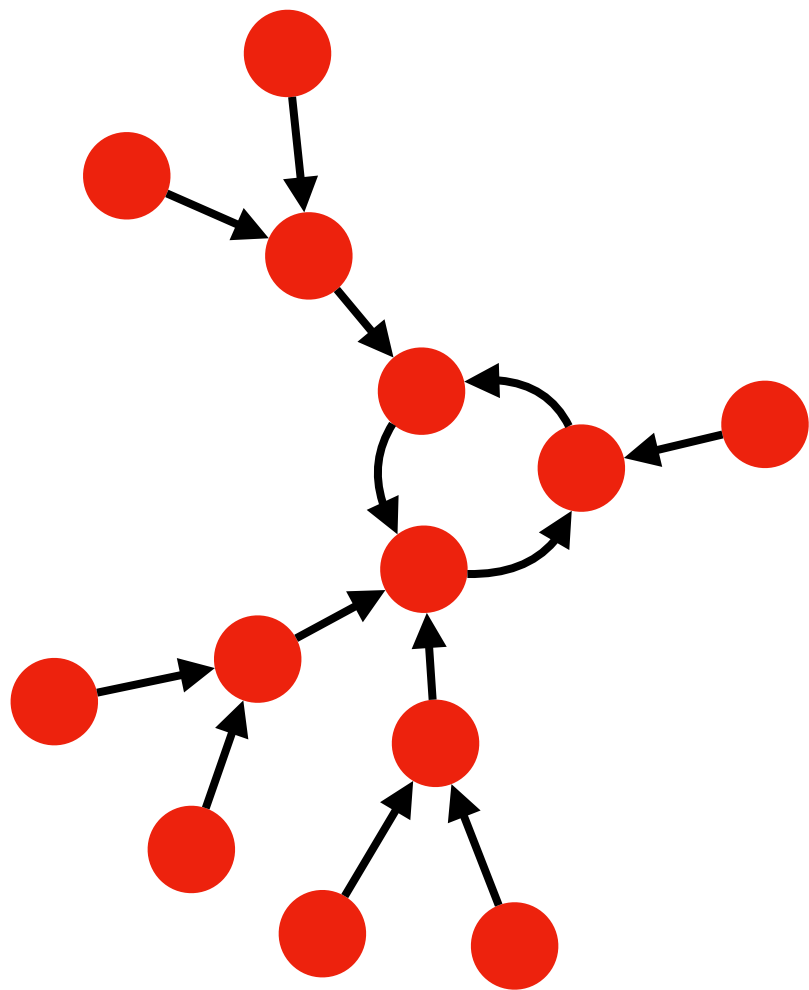
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A few limit cycles **with trees going in**



# General shape of a dynamical system

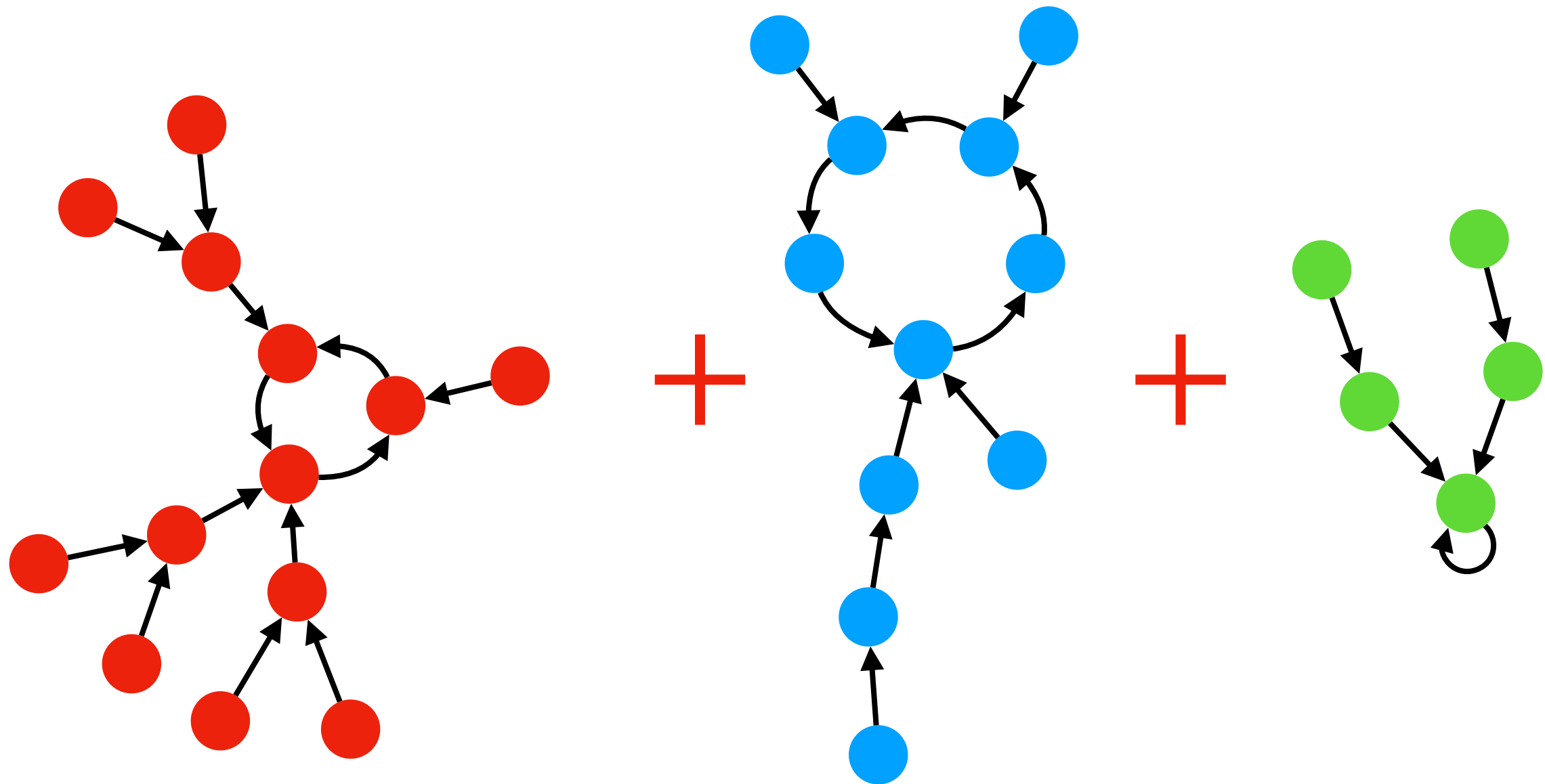
A few limit cycles **with trees going in**



$$C_3\left(\begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \bullet \bullet \\ \swarrow \downarrow \searrow \\ \bullet \bullet \bullet \end{array}, \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \downarrow \downarrow \\ \bullet \bullet \end{array}, \right) + C_5\left(\begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \bullet \bullet \bullet \\ \swarrow \downarrow \searrow \\ \bullet \bullet \bullet \end{array}, \bullet, \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \bullet\right) + C_1\left(\begin{array}{c} \bullet \\ \downarrow \downarrow \downarrow \\ \bullet \bullet \bullet \end{array}\right)$$

# General shape of a dynamical system

A few limit cycles **with trees going in**



$$C_3\left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}\right) + C_5\left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \bullet, \begin{array}{c} \bullet \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \end{array}, \bullet\right) + C_1\left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}\right)$$



# Not a problem, from a complexity perspective

2009 24th Annual IEEE Conference on Computational Complexity

# Planar Graph Isomorphism is in Log-Space

Samir Datta<sup>\*</sup>, Nutan Limaye<sup>†</sup>, Prajakta Nimbhorkar<sup>†</sup>, Thomas Thierauf<sup>‡</sup>, Fabian Wagner<sup>§</sup>  
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Email: fabian.wagner@uni-ulm.de

## Abstract

Graph Isomorphism is the prime example of a computational problem with a wide difference between the best known lower and upper bounds on its complexity. There is a significant gap between extant lower and upper bounds for planar graphs as well. We bridge the gap for this natural and important special case by presenting an upper bound and a lower bound on log-space hardness [JKMT03]. In

The problem is clearly in NP and by a group theoretic proof also in SPP [AK06]. This is the current frontier of our knowledge as far as upper bounds go. The inability to give efficient algorithms for the problem would lead one to believe that the problem is provably hard. NP-hardness is precluded by a result that states if GI is NP-hard then the polynomial time hierarchy collapses to the second level [BHZ87], [Sch88]. What is more surprising is that not even P-hardness is known for the problem. The best we know is that GI is hard for DET [Tor04], the class of problems reducible to the determinant, defined by Cook [Coo85].

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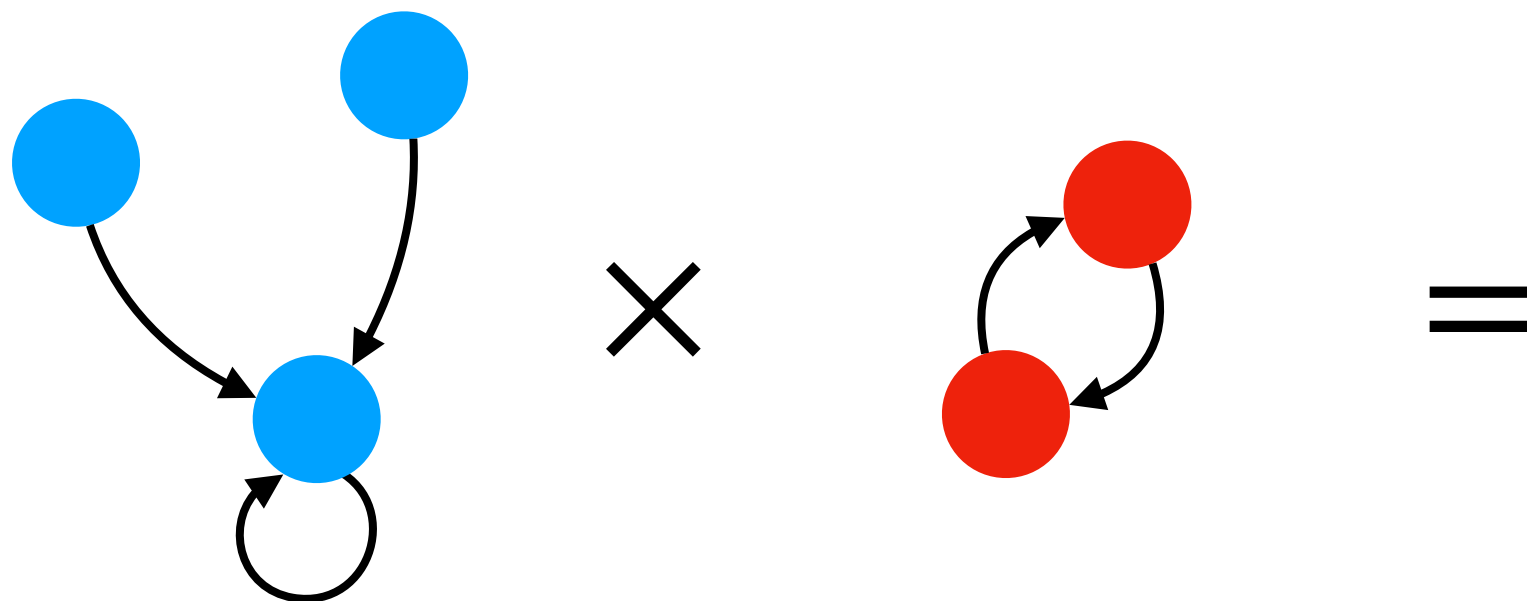
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Graph Isomorphism is the prime example of a computational problem with a wide difference between the best known lower and upper bounds on its complexity. There is a significant gap between extant lower and upper bounds for planar graphs as well. We bridge the gap for this natural and important special case by presenting an upper bound and a new log-space hardness [JKMT03]. In planar graph

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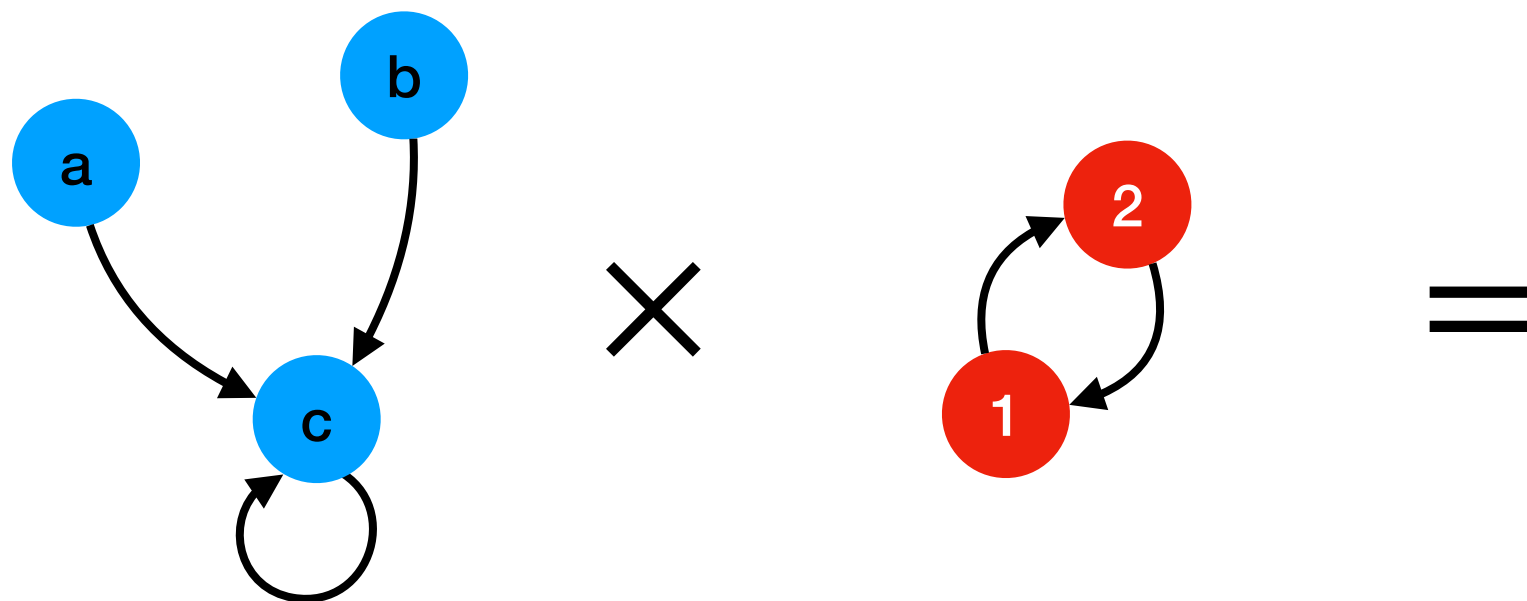
# Product is graph tensor product

Synchronous execution of two systems



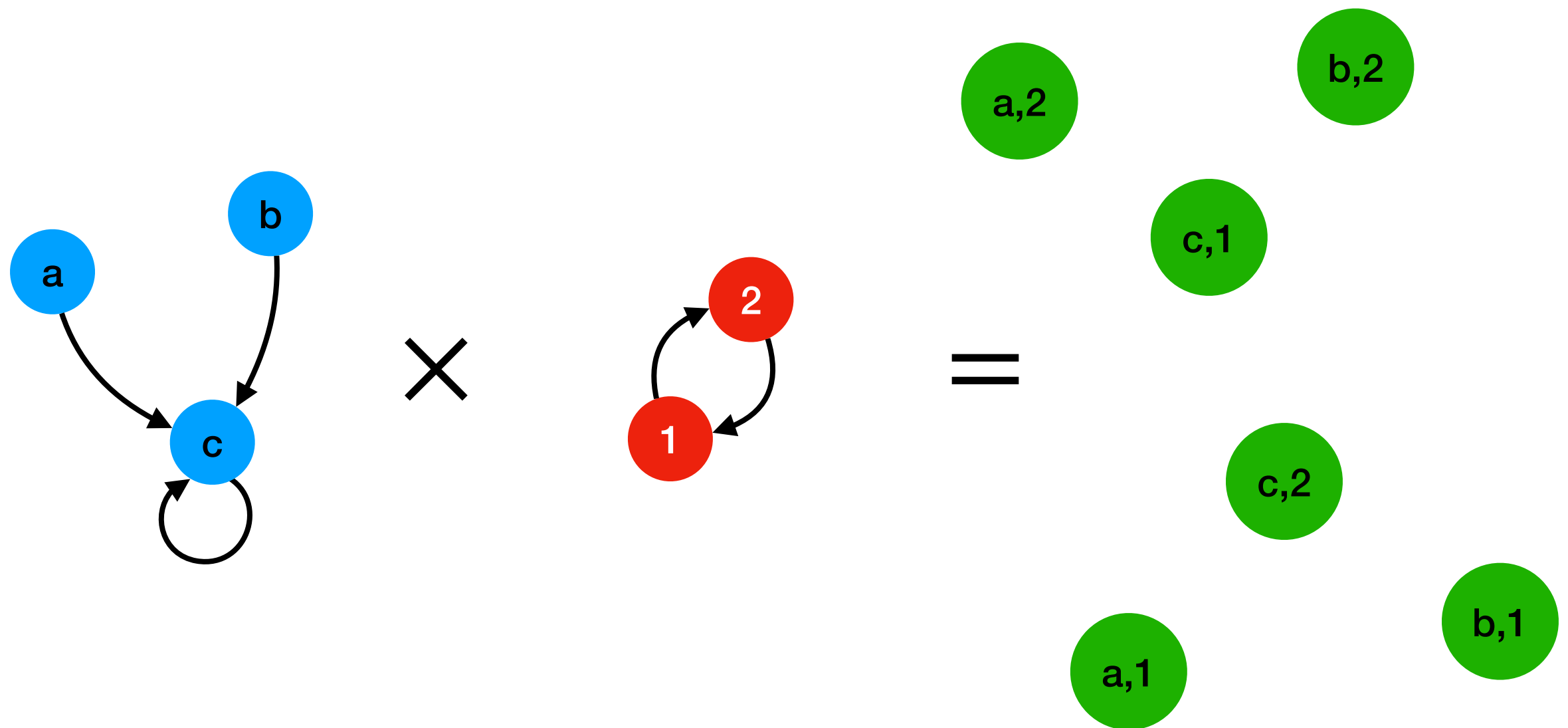
# Product in $\mathbf{D}$ is graph tensor product

Temporary state names



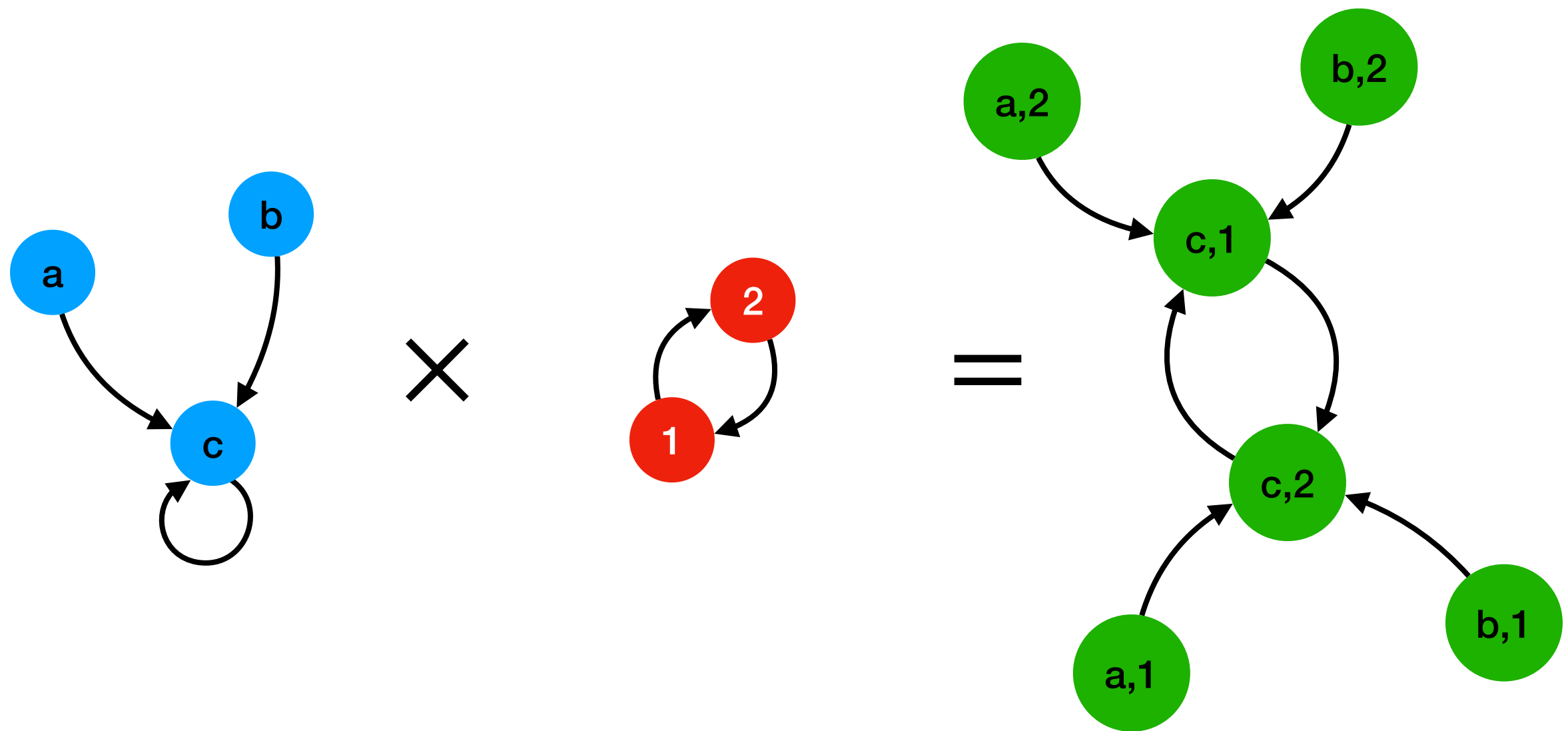
# Product in $\mathbf{D}$ is graph tensor product

## Cartesian product of the states



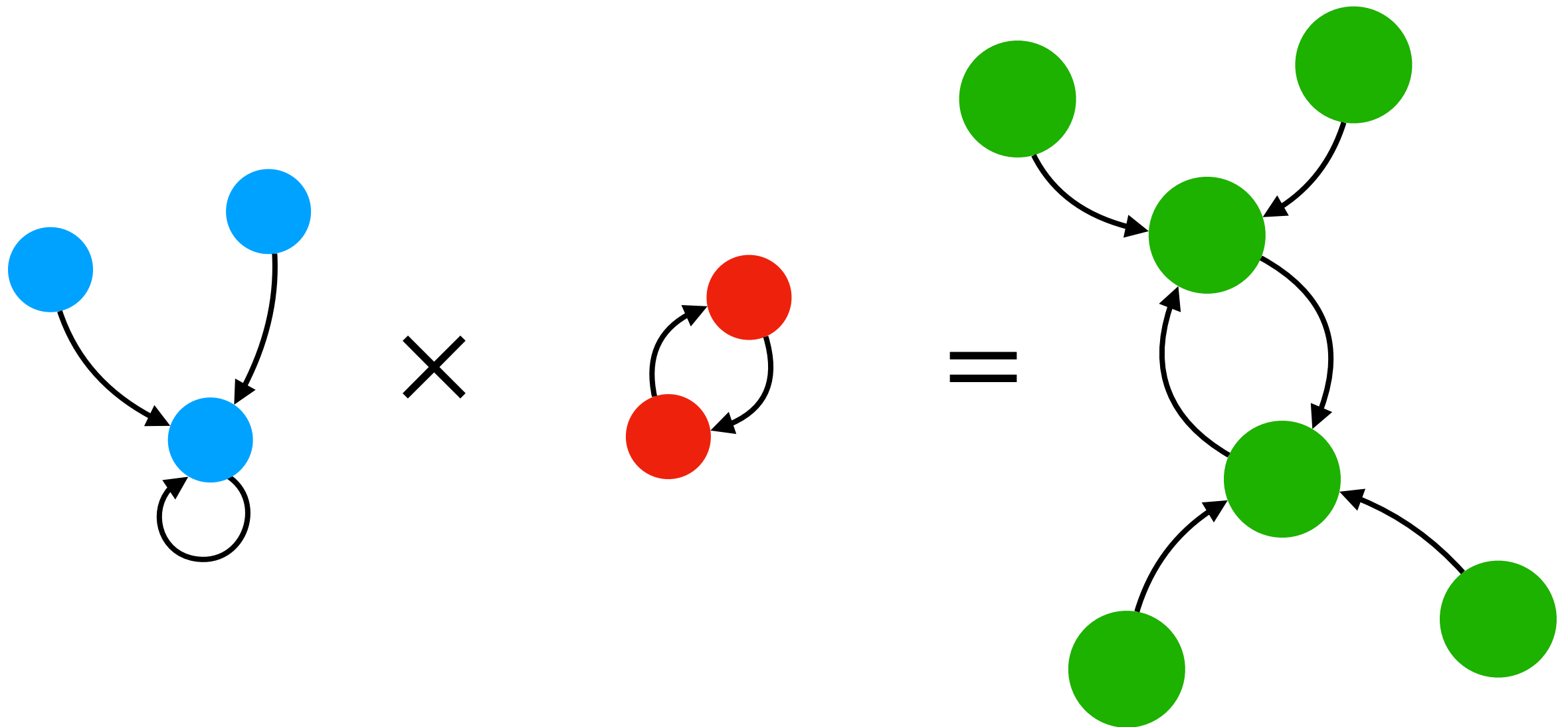
# Product in $\mathbf{D}$ is graph tensor product

Arrows iff arrows between both components



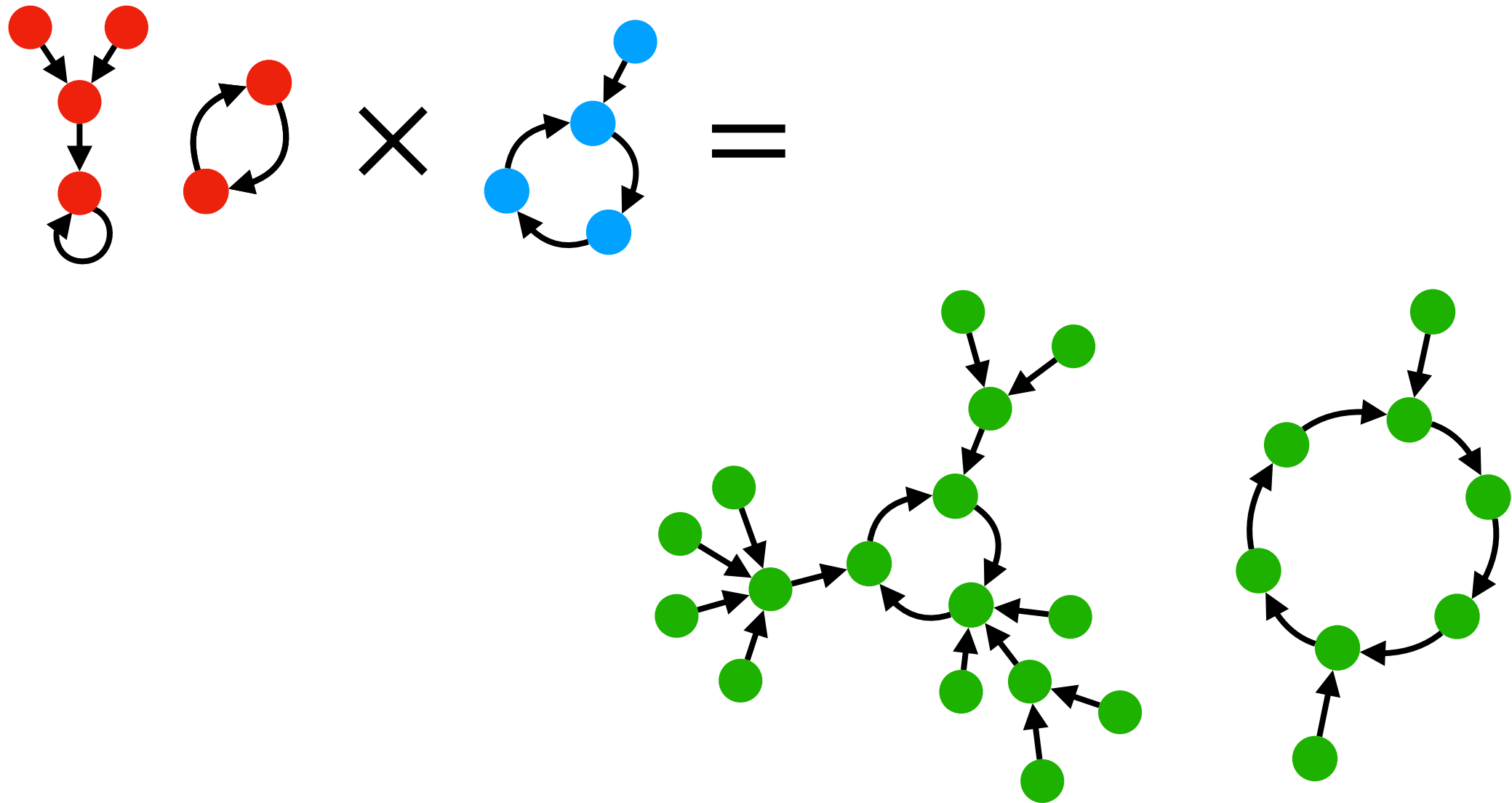
# Product in $\mathbf{D}$ is graph tensor product

We forget the state names once again



# Products “preserve” behaviours

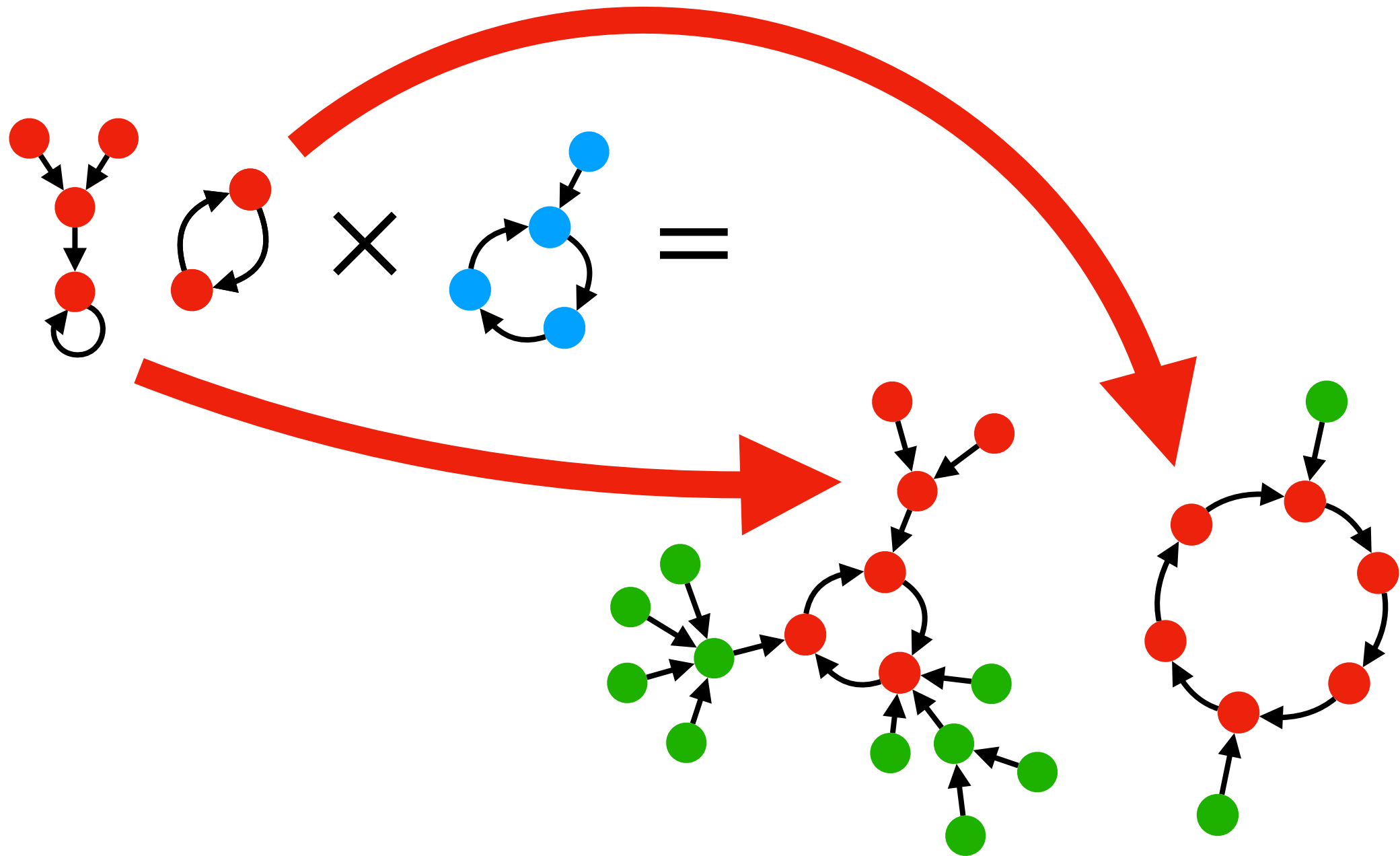
$A$  is a minor of  $A \times B$  for  $B \neq \emptyset$





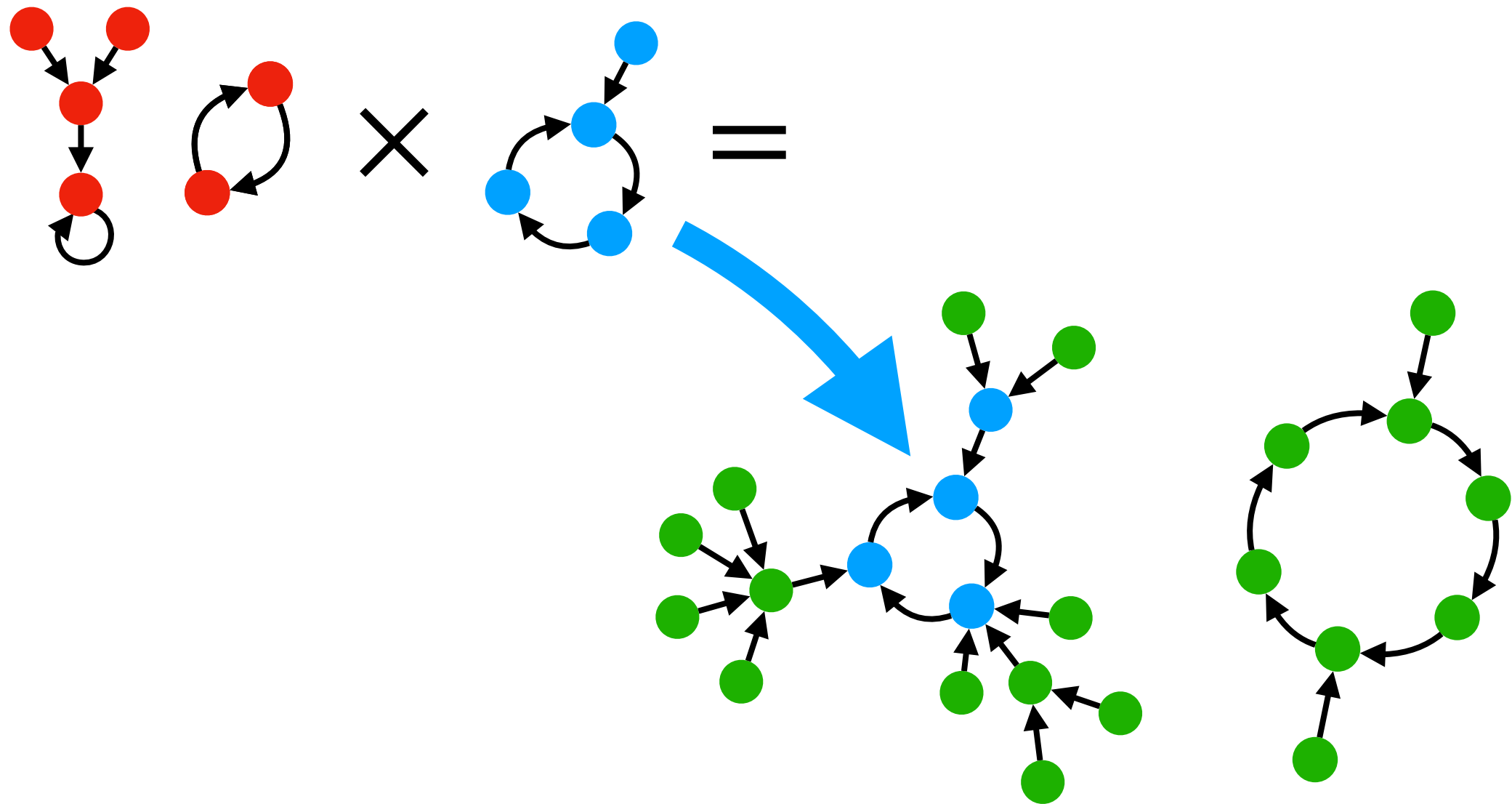
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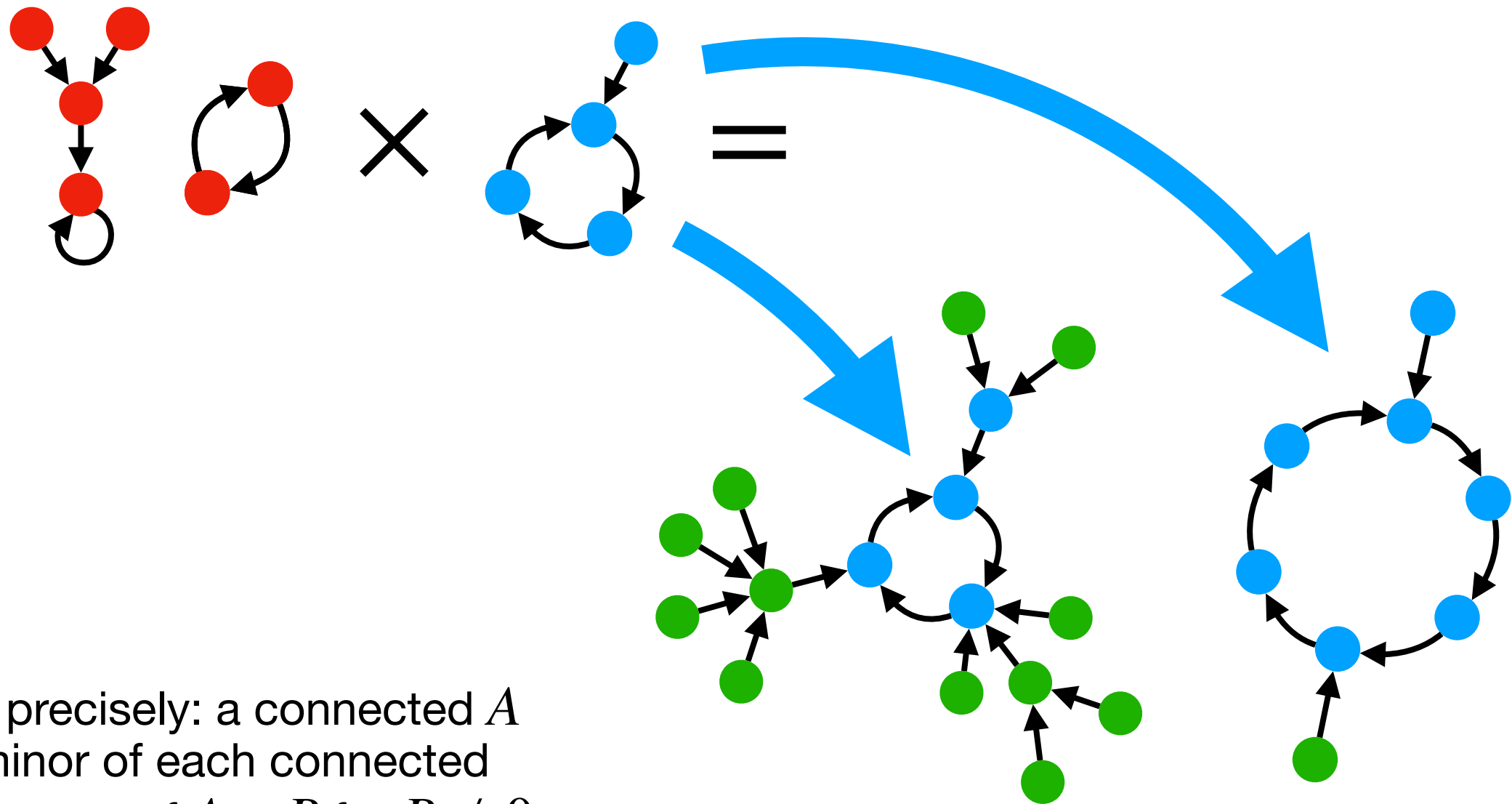
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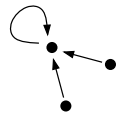
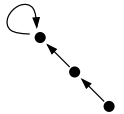














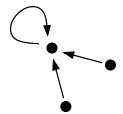
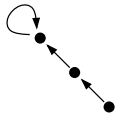


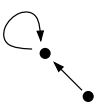
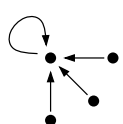
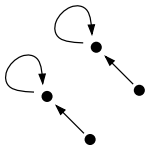
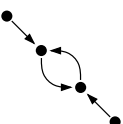
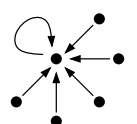
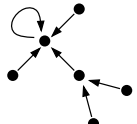
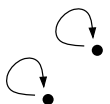

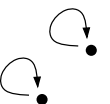
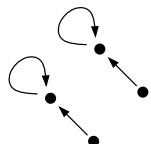
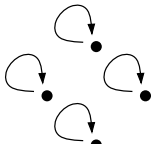
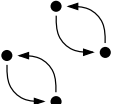
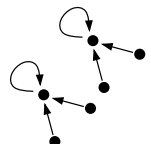
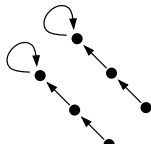



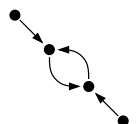
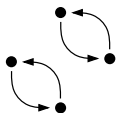

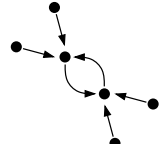
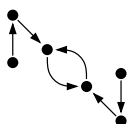
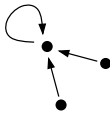

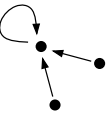
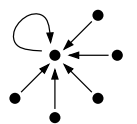
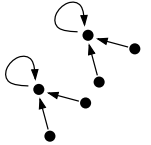
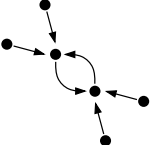
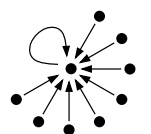
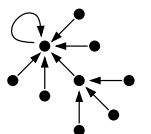
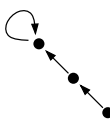

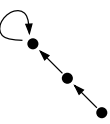
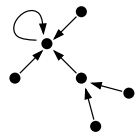
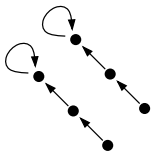
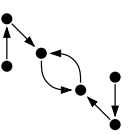
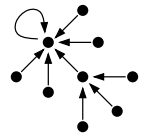
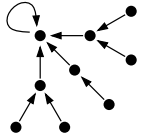
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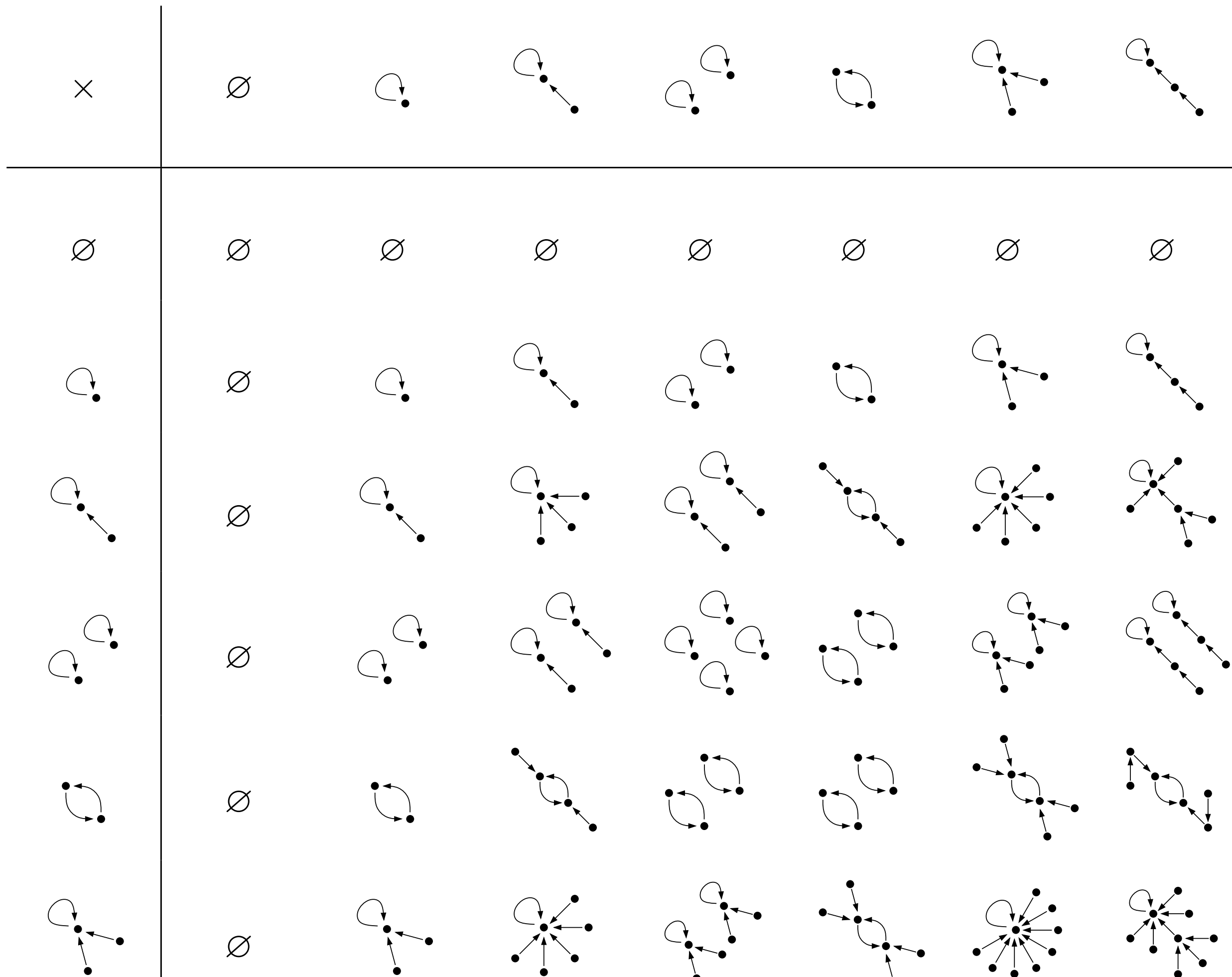



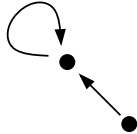
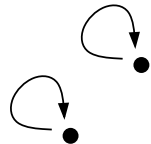

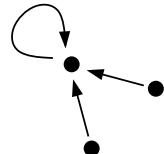
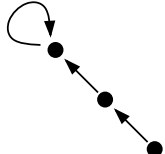


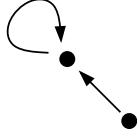
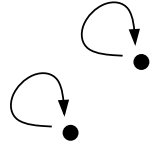

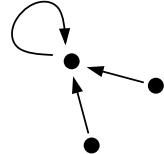
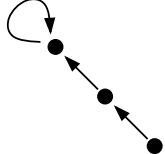
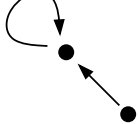
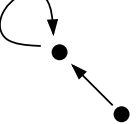
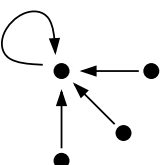
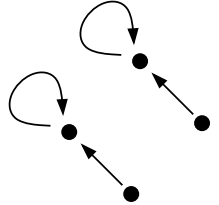
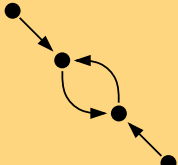
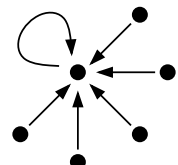
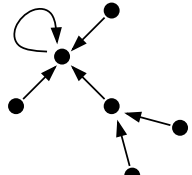
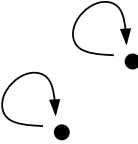
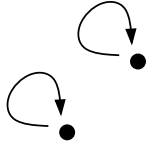
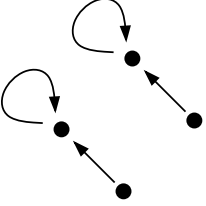
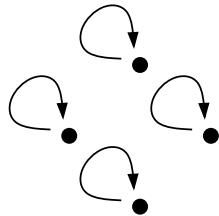
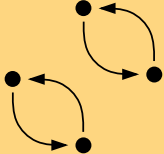
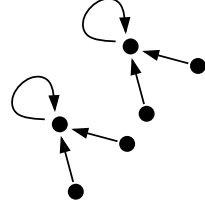
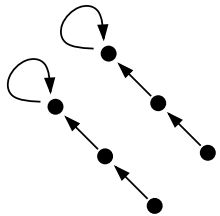


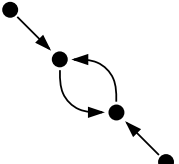
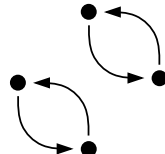
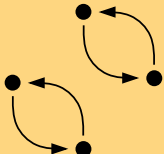
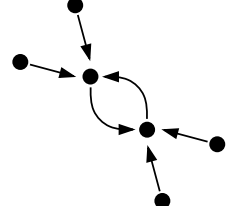
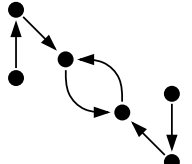
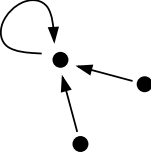
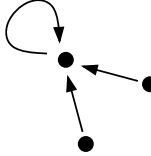
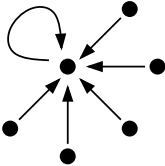
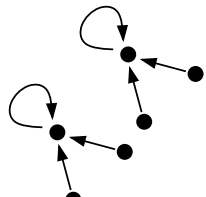
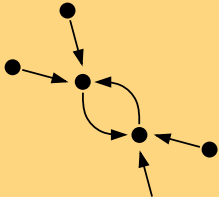
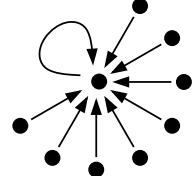
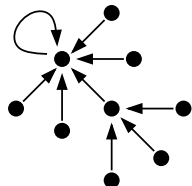
more precisely: a connected  $A$   
is a minor of each connected  
component of  $A \times B$  for  $B \neq \emptyset$



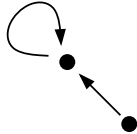
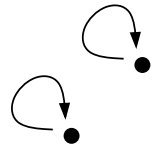

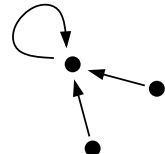
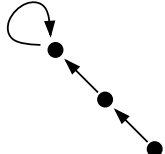











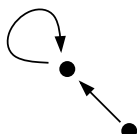
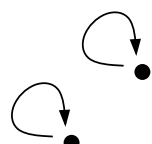

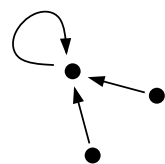
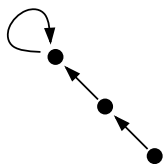
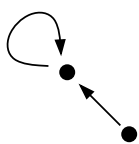

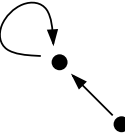
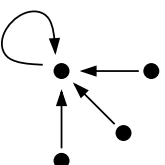
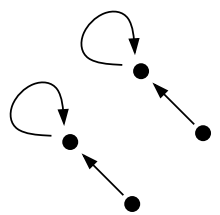
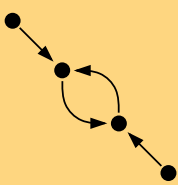
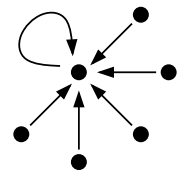
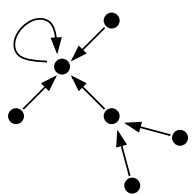
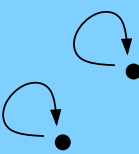

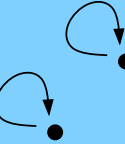
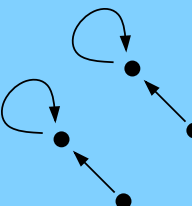
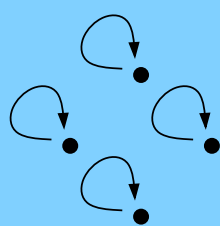
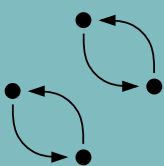
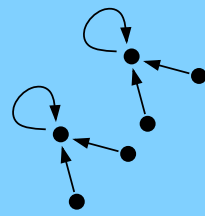
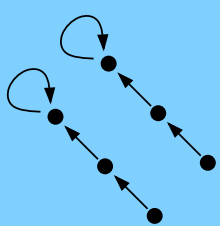



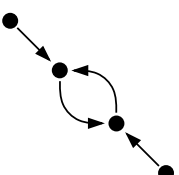
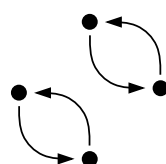
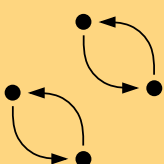
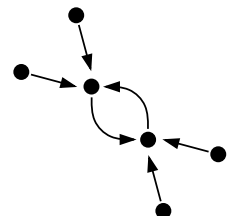
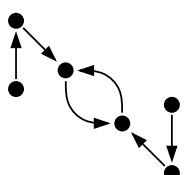
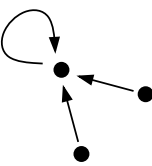

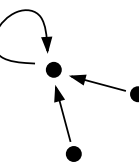
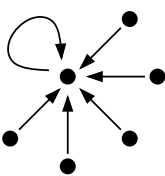
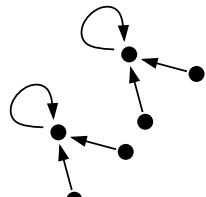
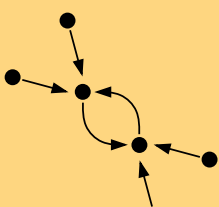
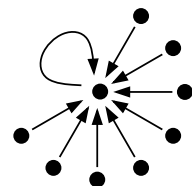
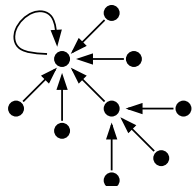
**No unique  
factorisation 🤔**

# Multiplication table


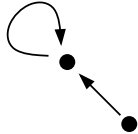
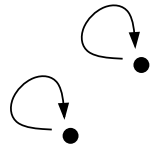

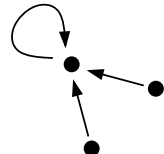
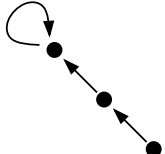


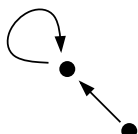
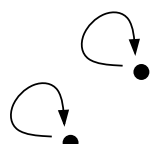
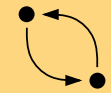
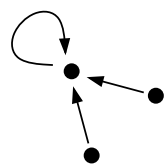
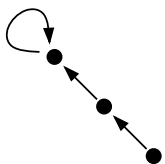
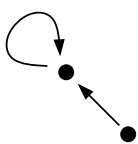
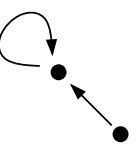
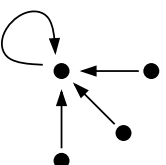
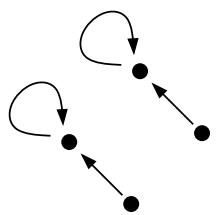
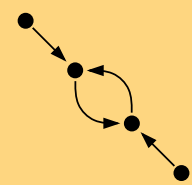
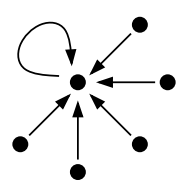
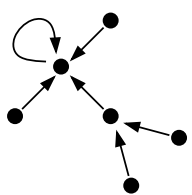
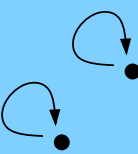
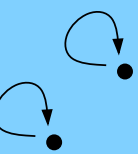
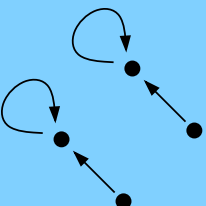
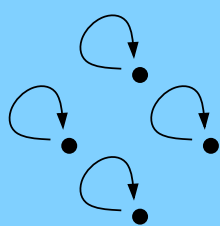
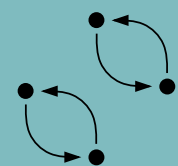
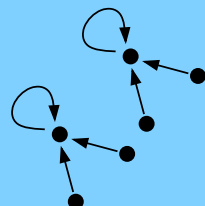
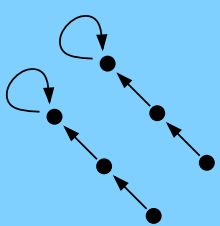


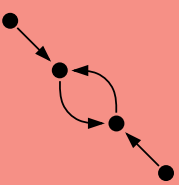
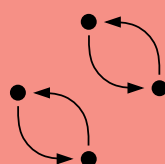
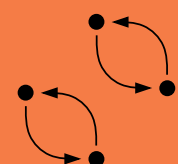
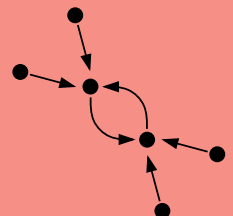
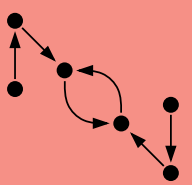
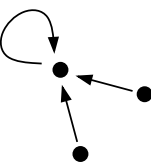
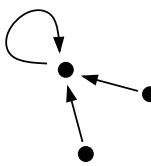
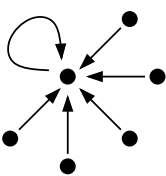
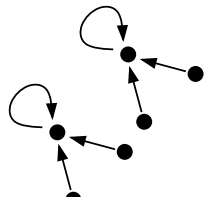
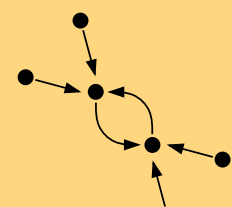
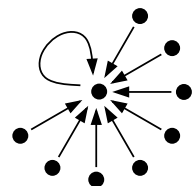
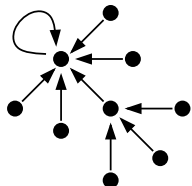
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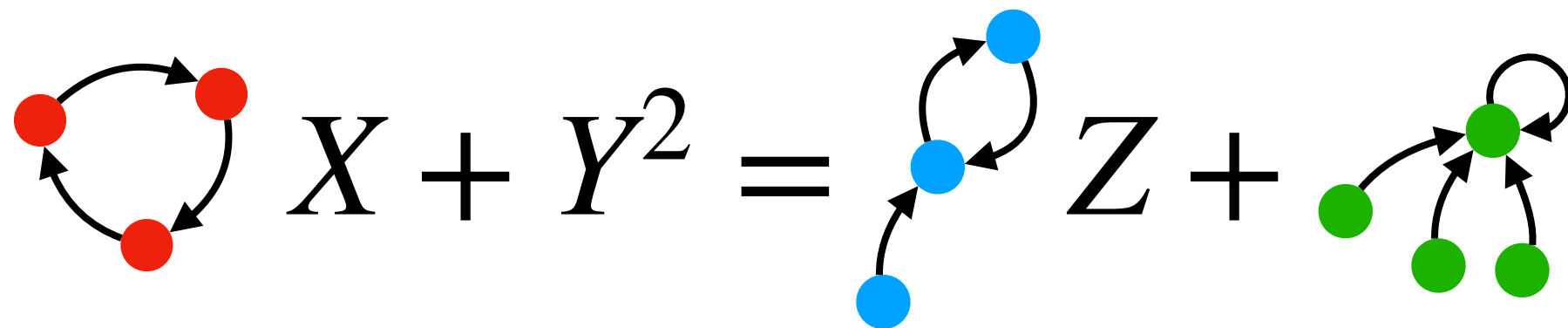
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# Polynomial equations

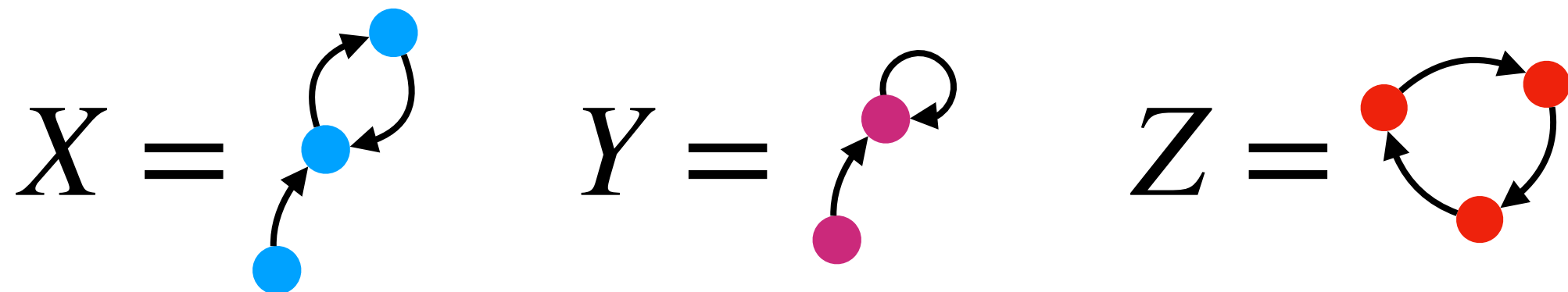
# Polynomial equations over D

For the analysis of complex systems

- Consider the equation



- There is least one solution



# Undecidability of polynomial equations

# $\mathbb{N}$ is a subsemiring of $\mathbf{D}$

This means trouble

- There is an injective homomorphism  $\varphi: \mathbb{N} \rightarrow \mathbf{D}$

$$\varphi(n) = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = \underbrace{\text{fixed point} + \text{fixed point} + \cdots + \text{fixed point}}_{n \text{ times}}$$

- $n$  fixed points behave exactly as the integer  $n$
- So  $\mathbf{D}$  contains an isomorphic copy of  $\mathbb{N}$

# Hilbert's 10th problem over $\mathbf{D}$

## Unsolvability of polynomial equations

- If a multivariate polynomial equation over  $\mathbb{N}$  has a solution in  $\mathbf{D}$ , then **it also has a solution in  $\mathbb{N}$**  (just replace each system by its size!)
- In the larger semiring  $\mathbf{D}$  we may find **extra solutions**, but only if the equation is **already solvable over the naturals**
- Then, by reduction from Hilbert's 10th problem, we obtain the undecidability in  $\mathbf{D}$  of **polynomial equations over  $\mathbb{N}$** ...
- ...and thus of arbitrary **equations over  $\mathbf{D}$**

**Polynomial equations  
with constant RHS are  
decidable and in NP**

**Systems of linear equations  
with constant RHS  
are NP-complete**



# NP-hardness of linear systems

## By reduction from One-in-three-3SAT

- Given a 3CNF Boolean formula  $\varphi$ , is there a satisfying assignment such that exactly **one literal per clause** is true?
- For **each variable  $x$  of  $\varphi$**  we have one equation  **$X + X' = 1$** , forcing one between  $X$  and  $X'$  to be 1, and the other to be 0
- For **each clause**, for instance  **$(x \vee \neg y \vee z)$** , we have one equation  **$X + Y' + Z = 1$** , which forces exactly one variable to 1
- These are all linear, constant-RHS equations over  **$\mathbf{D}$**  and more specifically over  $\mathbb{N}$ , and its **solutions are the same** as the satisfying assignments of  $\varphi$  with one true literal per clause

A **single** linear,  
constant-RHS equation  
is **NP**-complete\*

\* Main idea by Florian Bridoux, bravo !

# **$\mathbf{D}$ is a $\mathbb{N}$ -semimodule**

**Like a vector space, but over a semiring**

- Here the vectors are **dynamical systems** and the scalars are **naturals**
- Trivial because the semimodule axioms are a consequence of  $\mathbb{N}$  being a subsemiring of  $\mathbf{D}$
- $\mathbf{D}$  as a semimodule has a **unique, countably infinite basis** consisting of all **nonempty, connected dynamical systems**

# Reducing the system of equations to one

Several  $\mathbb{N}[\vec{X}]$  linear equations to one  $\mathbf{D}[\vec{X}]$  equation

- Let  $p_1(\vec{X}) = 1, \dots, p_n(\vec{X}) = 1$  be the previous system of equations, with  $p_i \in \mathbb{N}[\vec{X}]$
- Take any  $n$  easy-to-compute, linearly independent systems  $e_1, \dots, e_n \in \mathbf{D}$ , for instance

$$e_1 = \text{self-loop} \quad e_2 = \text{2-cycle} \quad e_3 = \text{3-cycle} \quad e_4 = \text{4-cycle} \quad \dots$$

- Then the equation  $e_1 p_1(\vec{X}) + \dots + e_n p_n(\vec{X}) = e_1 + \dots + e_n$  is a linear equation over  $\mathbf{D}[\vec{X}]$  having the same solutions as the original system

# Linear, constant-RHS eqns are NP-complete

## Even equations over cycles, even in explicit form!

### Reducing the system of equations to one

Several  $\mathbb{N}[\vec{X}]$  linear equations to one  $\mathbf{D}[\vec{X}]$  equation

- Let  $p_1(\vec{X}) = 1, \dots, p_n(\vec{X}) = 1$  be the previous system of equations, with  $p_i \in \mathbb{N}[\vec{X}]$
- Recall that  $\mathbf{D}$  is a  $\mathbb{N}$ -semimodule with basis all connected systems

- Take any  $n$  easy-to-compute, linearly independent systems

$$e_1 = \bullet \rightarrow \bullet$$

$$e_2 = \bullet \rightarrow \bullet \rightarrow \bullet$$

$$e_3 = \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$$

$$e_4 = \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$$

...

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**Irreducible *systems***

# Most dynamical systems are irreducible

**$A$  is irreducible iff  $A = BC$  implies  $B = 1$  or  $C = 1$**

- Formally:

$$\lim_{n \rightarrow \infty} \frac{\text{number of reducible systems over } \leq n \text{ states}}{\text{total number of systems over } \leq n \text{ states}} = 0$$


- Notice that this is **the opposite** of  $\mathbb{N}$ , where irreducible (aka prime) integers are scarce)

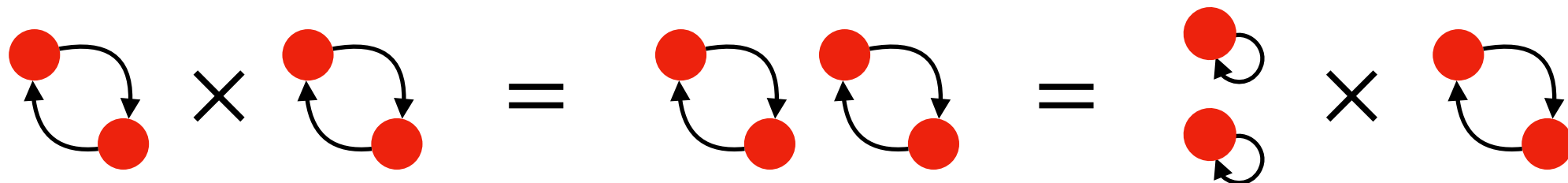


**Prime system**

# Prime system

$P \neq 0,1$  is prime iff  $P \mid AB$  implies  $P \mid A$  or  $P \mid B$

- If a prime  $P$  appears in a factorisation into irreducibles of a system, then **it appears in all factorisations**
- On the contrary, non-prime systems can sometimes be **replaced**
- So prime systems are **irreplaceable building blocks**
- We **don't know** if prime systems exist yet!
- But we know several nonprimes, for instance 



# More interesting classes of nonprimes

Work by Johan Couturier, bien joué !

- If  $A$  is **disconnected**, then  $A$  is not prime
- If  $A$  is connected but of **period**  $> 1$ , then  $A$  is not prime
- If  $A$  is connected of period 1, but

$$\gcd(A) = \gcd\{\#\text{preimages of } a : a \in A\} > 1$$

then  $A$  is not prime

- In particular, systems consisting of sums of cycles  
(i.e., **the asymptotic behaviours of any system**) are nonprime

# Is primality decidable?

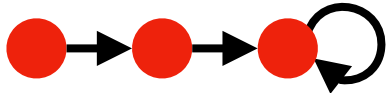
Most. Annoying. Open. Problem. Ever. 🤨

- We **do not know an algorithm** for primality testing!
- Nonprimes are **recursively enumerable**
  - Enumerate systems  $A, B$  to find a counterexample to the primality of  $P$ , i.e.,  $P \mid AB$  but  $P \nmid A$  and  $P \nmid B$
  - No known way to **bound the size** of counterexamples
- Fun fact: if primality is **undecidable**, then primes **do exist** 😊

# Open problems

# Open problems

## Algebraic ones

- Do **prime systems** exist at all? Is **primality decidable**?
- Is **this particular guy here** prime? 
- What is the complexity of **deciding if  $A \mid B$** ?  
And deciding if  **$A$  is irreducible**?
- Does it make any sense to **adjoin the additive inverses** in order to obtain a ring?
- Is it useful to find **nondeterministic dynamical system** (i.e., arbitrary graph) **solutions** to equations?
- Semirings of **infinite** discrete-time dynamical systems

# Open problems

## Solving equations

- Find **larger classes of solvable equations**,  
e.g., by number of variables or degree of the polynomials
- Discover classes of **equations solvable efficiently**
  - Probably very hard for systems in succinct form
- Find out if there exist **decidable equations harder than NP**
  - It would feel strange to jump from **NP** to undecidable

**Thanks for your attention!**  
**Merci de votre attention !**



***Any questions?***